

Standard Borel Spaces

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Abstract

This entry includes a formalization of standard Borel spaces and (a variant of) the Borel isomorphism theorem. A separable complete metrizable topological space is called a polish space and a measurable space generated from a polish space is called a standard Borel space. We formalize the notion of standard Borel spaces by establishing set-based metric spaces, and then prove (a variant of) the Borel isomorphism theorem. The theorem states that a standard Borel spaces is either a countable discrete space or isomorphic to $\mathbb{R}$.

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We refer to the HOL-Analysis library, the textbooks by Matsuzaka [2] and Srivastava [3], and the lecture note by Biskup [1].

1 Lemmas

```
theory Lemmas-StandardBorel
  imports HOL-Probability.Probability
begin
```

1.1 Lemmas for Abstract Topology

1.1.1 Generated By

```
lemma topology-generated-by-sub:
  assumes  $\bigwedge U. U \in \mathcal{U} \implies (\text{openin } X \ U)$ 
  and  $\text{openin } (\text{topology-generated-by } \mathcal{U}) \ U$ 
  shows  $\text{openin } X \ U$ 
  <proof>

lemma topology-generated-by-open:
   $S = \text{topology-generated-by } \{U \mid U . \text{openin } S \ U\}$ 
  <proof>

lemma topology-generated-by-eq:
  assumes  $\bigwedge U. U \in \mathcal{U} \implies (\text{openin } (\text{topology-generated-by } \mathcal{O}) \ U)$ 
  and  $\bigwedge U. U \in \mathcal{O} \implies (\text{openin } (\text{topology-generated-by } \mathcal{U}) \ U)$ 
  shows  $\text{topology-generated-by } \mathcal{O} = \text{topology-generated-by } \mathcal{U}$ 
  <proof>

lemma topology-generated-by-homeomorphic-spaces:
  assumes  $\text{homeomorphic-map } X \ Y \ f \ X = \text{topology-generated-by } \mathcal{O}$ 
  shows  $Y = \text{topology-generated-by } ((\cdot) \ f \ \mathcal{O})$ 
```

$\langle \text{proof} \rangle$

lemma *open-map-generated-topo:*

assumes $\bigwedge u. u \in U \implies \text{openin } S (f \circ u) \text{ inj-on } f \text{ (topspace (topology-generated-by } U))$

shows $\text{open-map (topology-generated-by } U) S f$

$\langle \text{proof} \rangle$

lemma *subtopology-generated-by:*

$\text{subtopology (topology-generated-by } \mathcal{O}) T = \text{topology-generated-by } \{T \cap U \mid U. U \in \mathcal{O}\}$

$\langle \text{proof} \rangle$

lemma *prod-topology-generated-by:*

$\text{topology-generated-by } \{U \times V \mid U V. U \in \mathcal{O} \wedge V \in \mathcal{U}\} = \text{prod-topology (topology-generated-by } \mathcal{O}) (\text{topology-generated-by } \mathcal{U})$

$\langle \text{proof} \rangle$

lemma *prod-topology-generated-by-open:*

$\text{prod-topology } S S' = \text{topology-generated-by } \{U \times V \mid U V. \text{openin } S U \wedge \text{openin } S' V\}$

$\langle \text{proof} \rangle$

lemma *product-topology-cong:*

assumes $\bigwedge i. i \in I \implies S i = K i$

shows $\text{product-topology } S I = \text{product-topology } K I$

$\langle \text{proof} \rangle$

lemma *topology-generated-by-without-empty:*

$\text{topology-generated-by } \mathcal{O} = \text{topology-generated-by } \{U \in \mathcal{O}. U \neq \{\}\}$

$\langle \text{proof} \rangle$

lemma *topology-from-bij:*

assumes $\text{bij-betw } f A \text{ (topspace } S)$

shows $\text{homeomorphic-map (pullback-topology } A f S) S f \text{topspace (pullback-topology } A f S) = A$

$\langle \text{proof} \rangle$

lemma *openin-pullback-topology':*

assumes $\text{bij-betw } f A \text{ (topspace } S)$

shows $\text{openin (pullback-topology } A f S) u \iff (\text{openin } S (f \circ u)) \wedge u \subseteq A$

$\langle \text{proof} \rangle$

1.1.2 Isolated Point

definition *isolated-points-of* :: 'a topology $\Rightarrow$ 'a set $\Rightarrow$ 'a set (**infixr** $\langle \text{'isolated'-points'-of} \rangle$ 80) **where**

$X \text{ isolated-points-of } A \equiv \{x \in \text{topspace } X \cap A. x \notin X \text{ derived-set-of } A\}$

lemma *isolated-points-of-eq*:

$X \text{ isolated-points-of } A = \{x \in \text{topspace } X \cap A. \exists U. x \in U \wedge \text{openin } X \ U \wedge U \cap (A - \{x\}) = \{\}\}$
 $\langle \text{proof} \rangle$

lemma *in-isolated-points-of*:

$x \in X \text{ isolated-points-of } A \longleftrightarrow x \in \text{topspace } X \wedge x \in A \wedge (\exists U. x \in U \wedge \text{openin } X \ U \wedge U \cap (A - \{x\}) = \{\})$
 $\langle \text{proof} \rangle$

lemma *derived-set-of-eq*:

$x \in X \text{ derived-set-of } A \longleftrightarrow x \in X \text{ closure-of } (A - \{x\})$
 $\langle \text{proof} \rangle$

1.1.3 Perfect Set

definition *perfect-set* :: 'a topology $\Rightarrow$ 'a set $\Rightarrow$ bool **where**

perfect-set $X \ A \longleftrightarrow \text{closedin } X \ A \wedge X \text{ isolated-points-of } A = \{\}$

abbreviation *perfect-space* $X \equiv \text{perfect-set } X \ (\text{topspace } X)$

lemma *perfect-space-euclidean*: *perfect-space* (*euclidean* :: 'a :: *perfect-space topology*)

$\langle \text{proof} \rangle$

lemma *perfect-setI*:

assumes *closedin* $X \ A$

and $\bigwedge x \ T. \llbracket x \in A; x \in T; \text{openin } X \ T \rrbracket \Longrightarrow \exists y \neq x. y \in T \wedge y \in A$

shows *perfect-set* $X \ A$

$\langle \text{proof} \rangle$

lemma *perfect-spaceI*:

assumes $\bigwedge x \ T. \llbracket x \in T; \text{openin } X \ T \rrbracket \Longrightarrow \exists y \neq x. y \in T$

shows *perfect-space* X

$\langle \text{proof} \rangle$

lemma *perfect-setD*:

assumes *perfect-set* $X \ A$

shows $\text{closedin } X \ A \subseteq \text{topspace } X \ \bigwedge x \ T. \llbracket x \in A; x \in T; \text{openin } X \ T \rrbracket \Longrightarrow \exists y \neq x. y \in T \wedge y \in A$

$\langle \text{proof} \rangle$

lemma *perfect-space-perfect*:

perfect-set euclidean (*UNIV* :: 'a :: *perfect-space set*)

$\langle \text{proof} \rangle$

lemma *perfect-set-subtopology*:

assumes *perfect-set* $X \ A$

shows *perfect-space* (*subtopology* $X \ A$)

$\langle proof \rangle$

1.1.4 Bases and Sub-Bases in Abstract Topology

definition *subbase-in* :: [*'a topology*, *'a set set*] $\Rightarrow$ *bool* **where**
subbase-in *S* $\mathcal{O} \longleftrightarrow S = \text{topology-generated-by } \mathcal{O}$

definition *base-in* :: [*'a topology*, *'a set set*] $\Rightarrow$ *bool* **where**
base-in *S* $\mathcal{O} \longleftrightarrow (\forall U. \text{openin } S \ U \longleftrightarrow (\exists \mathcal{U}. U = \bigcup \mathcal{U} \wedge \mathcal{U} \subseteq \mathcal{O}))$

lemma *second-countable-base-in*: *second-countable* *S* $\longleftrightarrow (\exists \mathcal{O}. \text{countable } \mathcal{O} \wedge \text{base-in } S \ \mathcal{O})$
 $\langle proof \rangle$

definition *zero-dimensional* :: *'a topology* $\Rightarrow$ *bool* **where**
zero-dimensional *S* $\longleftrightarrow (\exists \mathcal{O}. \text{base-in } S \ \mathcal{O} \wedge (\forall u \in \mathcal{O}. \text{openin } S \ u \wedge \text{closedin } S \ u))$

lemma *openin-base*:
 assumes *base-in* *S* $\mathcal{O}$ $U = \bigcup \mathcal{U}$ **and** $\mathcal{U} \subseteq \mathcal{O}$
 shows *openin* *S* *U*
 $\langle proof \rangle$

lemma *base-is-subbase*:
 assumes *base-in* *S* $\mathcal{O}$
 shows *subbase-in* *S* $\mathcal{O}$
 $\langle proof \rangle$

lemma *subbase-in-subset*:
 assumes *subbase-in* *S* $\mathcal{O}$ **and** $U \in \mathcal{O}$
 shows $U \subseteq \text{topspace } S$
 $\langle proof \rangle$

lemma *subbase-in-openin*:
 assumes *subbase-in* *S* $\mathcal{O}$ **and** $U \in \mathcal{O}$
 shows *openin* *S* *U*
 $\langle proof \rangle$

lemma *base-in-subset*:
 assumes *base-in* *S* $\mathcal{O}$ **and** $U \in \mathcal{O}$
 shows $U \subseteq \text{topspace } S$
 $\langle proof \rangle$

lemma *base-in-openin*:
 assumes *base-in* *S* $\mathcal{O}$ **and** $U \in \mathcal{O}$
 shows *openin* *S* *U*
 $\langle proof \rangle$

lemma *base-in-def2*:
 assumes $\bigwedge U. U \in \mathcal{O} \implies \text{openin } S \ U$

shows $\text{base-in } S \mathcal{O} \longleftrightarrow (\forall U. \text{openin } S U \longrightarrow (\forall x \in U. \exists W \in \mathcal{O}. x \in W \wedge W \subseteq U))$
 $\langle \text{proof} \rangle$

lemma base-in-def2' :
 $\text{base-in } S \mathcal{O} \longleftrightarrow (\forall b \in \mathcal{O}. \text{openin } S b) \wedge (\forall x. \text{openin } S x \longrightarrow (\exists B' \subseteq \mathcal{O}. \bigcup B' = x))$
 $\langle \text{proof} \rangle$

corollary base-in-in-subset :
assumes $\text{base-in } S \mathcal{O} \text{ openin } S u \ x \in u$
shows $\exists v \in \mathcal{O}. x \in v \wedge v \subseteq u$
 $\langle \text{proof} \rangle$

lemma $\text{base-in-without-empty}$:
assumes $\text{base-in } S \mathcal{O}$
shows $\text{base-in } S \{U \in \mathcal{O}. U \neq \{\}\}$
 $\langle \text{proof} \rangle$

lemma $\text{second-countable-ex-without-empty}$:
assumes $\text{second-countable } S$
shows $\exists \mathcal{O}. \text{countable } \mathcal{O} \wedge \text{base-in } S \mathcal{O} \wedge (\forall U \in \mathcal{O}. U \neq \{\})$
 $\langle \text{proof} \rangle$

lemma $\text{subtopology-subbase-in}$:
assumes $\text{subbase-in } S \mathcal{O}$
shows $\text{subbase-in } (\text{subtopology } S T) \{T \cap U \mid U. U \in \mathcal{O}\}$
 $\langle \text{proof} \rangle$

lemma $\text{subtopology-base-in}$:
assumes $\text{base-in } S \mathcal{O}$
shows $\text{base-in } (\text{subtopology } S T) \{T \cap U \mid U. U \in \mathcal{O}\}$
 $\langle \text{proof} \rangle$

lemma $\text{second-countable-subtopology}$:
assumes $\text{second-countable } S$
shows $\text{second-countable } (\text{subtopology } S T)$
 $\langle \text{proof} \rangle$

lemma $\text{open-map-with-base}$:
assumes $\text{base-in } S \mathcal{O} \bigwedge A. A \in \mathcal{O} \implies \text{openin } S' (f \text{ ` } A)$
shows $\text{open-map } S S' f$
 $\langle \text{proof} \rangle$

Construct a base from a subbase.

lemma $\text{finite'-intersection-of-idempot [simp]}$:
 $\text{finite' intersection-of finite' intersection-of } P = \text{finite' intersection-of } P$
 $\langle \text{proof} \rangle$

lemma *finite'-intersection-of-countable*:

assumes *countable* $\mathcal{O}$

shows *countable* (*Collect* (*finite' intersection-of* ($\lambda x. x \in \mathcal{O}$)))

<proof>

lemma *finite'-intersection-of-openin*:

assumes (*finite' intersection-of* ($\lambda x. x \in \mathcal{O}$)) *U*

shows *openin* (*topology-generated-by* $\mathcal{O}$) *U*

<proof>

lemma *topology-generated-by-finite-intersections*:

topology-generated-by $\mathcal{O} = \text{topology-generated-by } (\text{Collect } (\text{finite' intersection-of } (\lambda x. x \in \mathcal{O})))$

<proof>

lemma *base-from-subbase*:

assumes *subbase-in* $S \mathcal{O}$

shows *base-in* S (*Collect* (*finite' intersection-of* ($\lambda x. x \in \mathcal{O}$)))

<proof>

lemma *countable-base-from-countable-subbase*:

assumes *countable* $\mathcal{O}$ **and** *subbase-in* $S \mathcal{O}$

shows *second-countable* S

<proof>

lemma *prod-topology-second-countable*:

assumes *second-countable* S **and** *second-countable* S'

shows *second-countable* (*prod-topology* $S S'$)

<proof>

Abstract version of the theorem $\exists K. \text{topological-basis } K \wedge \text{countable } K \wedge (\forall k \in K. \exists X. k = \text{Pi}_E \text{ UNIV } X \wedge (\forall i. \text{open } (X i)) \wedge \text{finite } \{i. X i \neq \text{UNIV}\})$.

lemma *product-topology-countable-base-in*:

assumes *countable* I **and** $\bigwedge i. i \in I \implies \text{second-countable } (S i)$

shows $\exists \mathcal{O}'. \text{countable } \mathcal{O}' \wedge \text{base-in } (\text{product-topology } S I) \mathcal{O}' \wedge$

$(\forall k \in \mathcal{O}'. \exists X. k = (\text{Pi}_E i \in I. X i) \wedge (\forall i. \text{openin } (S i) (X i)) \wedge \text{finite } \{i. X i \neq \text{topspace } (S i)\} \wedge \{i. X i \neq \text{topspace } (S i)\} \subseteq I)$

<proof>

lemma *product-topology-second-countable*:

assumes *countable* I **and** $\bigwedge i. i \in I \implies \text{second-countable } (S i)$

shows *second-countable* (*product-topology* $S I$)

<proof>

lemma *second-countable-euclidean[simp]*:

second-countable (*euclidean* :: 'a :: *second-countable-topology topology*)

<proof>

lemma *Cantor-Bendixon*:

assumes *second-countable X*

shows $\exists U P. \text{countable } U \wedge \text{openin } X U \wedge \text{perfect-set } X P \wedge U \cup P = \text{topspace } X \wedge U \cap P = \{\} \wedge (\forall a \neq \{\}. \text{openin } (\text{subtopology } X P) a \longrightarrow \text{uncountable } a)$
 $\langle \text{proof} \rangle$

1.1.5 Separable Spaces

definition *dense-in* :: $['a \text{ topology}, 'a \text{ set}] \Rightarrow \text{bool}$ **where**

dense-in S U $\longleftrightarrow (U \subseteq \text{topspace } S \wedge (\forall V. \text{openin } S V \longrightarrow V \neq \{\} \longrightarrow U \cap V \neq \{\}))$

lemma *dense-in-def2*:

dense-in S U $\longleftrightarrow (U \subseteq \text{topspace } S \wedge (S \text{ closure-of } U) = \text{topspace } S)$
 $\langle \text{proof} \rangle$

lemma *dense-in-topspace[simp]*: *dense-in S (topspace S)*

$\langle \text{proof} \rangle$

lemma *dense-in-subset*:

assumes *dense-in S U*

shows $U \subseteq \text{topspace } S$

$\langle \text{proof} \rangle$

lemma *dense-in-nonempty*:

assumes $\text{topspace } S \neq \{\}$ *dense-in S U*

shows $U \neq \{\}$

$\langle \text{proof} \rangle$

lemma *dense-inI*:

assumes $U \subseteq \text{topspace } S$

and $\bigwedge V. \text{openin } S V \Longrightarrow V \neq \{\} \Longrightarrow U \cap V \neq \{\}$

shows *dense-in S U*

$\langle \text{proof} \rangle$

lemma *dense-in-infinite*:

assumes *t1-space X infinite (topspace X) dense-in X U*

shows *infinite U*

$\langle \text{proof} \rangle$

lemma *dense-in-prod*:

assumes *dense-in S U* **and** *dense-in S' U'*

shows *dense-in (prod-topology S S') (U $\times$ U')*

$\langle \text{proof} \rangle$

lemma *separable-space-def2*: *separable-space S* $\longleftrightarrow (\exists U. \text{countable } U \wedge \text{dense-in } S U)$

$\langle \text{proof} \rangle$

lemma *countable-space-separable-space:*

assumes *countable* (*topspace S*)

shows *separable-space S*

<proof>

lemma *separable-space-prod:*

assumes *separable-space S and separable-space S'*

shows *separable-space (prod-topology S S')*

<proof>

lemma *dense-in-product:*

assumes $\bigwedge i. i \in I \implies \text{dense-in } (T\ i) \ (U\ i)$

shows *dense-in (product-topology T I) ($\Pi_{E\ i \in I}. U\ i$)*

<proof>

lemma *separable-countable-product:*

assumes *countable I and* $\bigwedge i. i \in I \implies \text{separable-space } (T\ i)$

shows *separable-space (product-topology T I)*

<proof>

lemma *separable-finite-product:*

assumes *finite I and* $\bigwedge i. i \in I \implies \text{separable-space } (T\ i)$

shows *separable-space (product-topology T I)*

<proof>

1.1.6 G_δ Set

lemma *gdelta-inD:*

assumes *gdelta-in S A*

shows $\exists \mathcal{U}. \mathcal{U} \neq \{\} \wedge \text{countable } \mathcal{U} \wedge (\forall b \in \mathcal{U}. \text{openin } S\ b) \wedge A = \bigcap \mathcal{U}$

<proof>

lemma *gdelta-inD':*

assumes *gdelta-in S A*

shows $\exists U. (\forall n :: \text{nat}. \text{openin } S\ (U\ n)) \wedge A = \bigcap (\text{range } U)$

<proof>

lemma *gdelta-in-continuous-map:*

assumes *continuous-map X Y f gdelta-in Y a*

shows *gdelta-in X (f $\circ$ a $\cap$ topspace X)*

<proof>

lemma *g-delta-of-inj-open-map:*

assumes *open-map X Y f inj-on f (topspace X) gdelta-in X a*

shows *gdelta-in Y (f $\circ$ a)*

<proof>

lemma *gdelta-in-prod:*

assumes *gdelta-in X A gdelta-in Y B*

shows *gdelta-in* (*prod-topology* $X\ Y$) ($A \times B$)
 <proof>

corollary *gdelta-in-prod1*:
assumes *gdelta-in* $X\ A$
shows *gdelta-in* (*prod-topology* $X\ Y$) ($A \times \text{topspace } Y$)
 <proof>

corollary *gdelta-in-prod2*:
assumes *gdelta-in* $Y\ B$
shows *gdelta-in* (*prod-topology* $X\ Y$) ($\text{topspace } X \times B$)
 <proof>

lemma *continuous-map-imp-closed-graph'*:
assumes *continuous-map* $X\ Y\ f$ *Hausdorff-space* Y
shows *closedin* (*prod-topology* $Y\ X$) ($(\lambda x. (f\ x, x)) \text{ ' } \text{topspace } X$)
 <proof>

1.1.7 Continuous Maps on First Countable Topology

Generalized version of *Metric-space* $?M\ ?d \implies \text{eventually } ?P\ (\text{atin } (\text{Metric-space.mtopology } ?M\ ?d)\ ?a) = (\forall \sigma. \text{range } \sigma \subseteq ?M - \{?a\} \wedge \text{limitin } (\text{Metric-space.mtopology } ?M\ ?d)\ \sigma\ ?a \text{ sequentially} \longrightarrow (\forall_F n \text{ in sequentially. } ?P\ (\sigma\ n)))$

lemma *eventually-atin-sequentially*:
assumes *first-countable* X
shows *eventually* $P\ (\text{atin } X\ a) \longleftrightarrow (\forall \sigma. \text{range } \sigma \subseteq \text{topspace } X - \{a\} \wedge \text{limitin } X\ \sigma\ a \text{ sequentially} \longrightarrow \text{eventually } (\lambda n. P\ (\sigma\ n)) \text{ sequentially})$
 <proof>

lemma *continuous-map-iff-limit-seq*:
assumes *first-countable* X
shows *continuous-map* $X\ Y\ f \longleftrightarrow (\forall xn\ x. \text{limitin } X\ xn\ x \text{ sequentially} \longrightarrow \text{limitin } Y\ (\lambda n. f\ (xn\ n))\ (f\ x) \text{ sequentially})$
 <proof>

1.1.8 Upper-Semicontinuous Functions

definition *upper-semicontinuous-map* :: [$'a\ \text{topology},\ 'a \Rightarrow 'b :: \text{linorder-topology}$]
 $\Rightarrow \text{bool}$ **where**
upper-semicontinuous-map $X\ f \longleftrightarrow (\forall a. \text{openin } X\ \{x \in \text{topspace } X. f\ x < a\})$

lemma *continuous-upper-semicontinuous*:
assumes *continuous-map* $X\ (\text{euclidean} :: ('b :: \text{linorder-topology})\ \text{topology})\ f$
shows *upper-semicontinuous-map* $X\ f$
 <proof>

lemma *upper-semicontinuous-map-iff-closed*:
upper-semicontinuous-map $X\ f \longleftrightarrow (\forall a. \text{closedin } X\ \{x \in \text{topspace } X. f\ x \geq a\})$
 <proof>

lemma *upper-semicontinuous-map-real-iff*:
fixes $f :: 'a \Rightarrow \text{real}$
shows $\text{upper-semicontinuous-map } X f \longleftrightarrow \text{upper-semicontinuous-map } X (\lambda x. \text{ereal } (f x))$
 $\langle \text{proof} \rangle$

1.1.9 Lower-Semicontinuous Functions

definition *lower-semicontinuous-map* :: $['a \text{ topology}, 'a \Rightarrow 'b :: \text{linorder-topology}] \Rightarrow \text{bool}$ **where**
 $\text{lower-semicontinuous-map } X f \longleftrightarrow (\forall a. \text{openin } X \{x \in \text{topspace } X. a < f x\})$

lemma *continuous-lower-semicontinuous*:
assumes $\text{continuous-map } X (\text{euclidean} :: ('b :: \text{linorder-topology}) \text{ topology}) f$
shows $\text{lower-semicontinuous-map } X f$
 $\langle \text{proof} \rangle$

lemma *lower-semicontinuous-map-iff-closed*:
 $\text{lower-semicontinuous-map } X f \longleftrightarrow (\forall a. \text{closedin } X \{x \in \text{topspace } X. f x \leq a\})$
 $\langle \text{proof} \rangle$

lemma *lower-semicontinuous-map-real-iff*:
fixes $f :: 'a \Rightarrow \text{real}$
shows $\text{lower-semicontinuous-map } X f \longleftrightarrow \text{lower-semicontinuous-map } X (\lambda x. \text{ereal } (f x))$
 $\langle \text{proof} \rangle$

1.2 Lemmas for Measure Theory

1.2.1 Lemmas for Measurable Sets

lemma *measurable-preserve-sigma-sets*:
assumes $\text{sets } M = \text{sigma-sets } \Omega \text{ } S \text{ } S \subseteq \text{Pow } \Omega$
 $\bigwedge a. a \in S \implies f ' a \in \text{sets } N \text{ inj-on } f (\text{space } M) f ' \text{space } M \in \text{sets } N$
and $b \in \text{sets } M$
shows $f ' b \in \text{sets } N$
 $\langle \text{proof} \rangle$

inductive-set *sigma-sets-cinter* :: $'a \text{ set} \Rightarrow 'a \text{ set set} \Rightarrow 'a \text{ set set}$
for $sp :: 'a \text{ set}$ **and** $A :: 'a \text{ set set}$
where
 $\text{Basic-c}[\text{intro}, \text{simp}]: a \in A \implies a \in \text{sigma-sets-cinter } sp A$
 $| \text{Top-c}[\text{simp}]: sp \in \text{sigma-sets-cinter } sp A$
 $| \text{Inter-c}: (\bigwedge i::\text{nat}. a i \in \text{sigma-sets-cinter } sp A) \implies (\bigcap i. a i) \in \text{sigma-sets-cinter } sp A$
 $| \text{Union-c}: (\bigwedge i::\text{nat}. a i \in \text{sigma-sets-cinter } sp A) \implies (\bigcup i. a i) \in \text{sigma-sets-cinter } sp A$

inductive-set *sigma-sets-cinter-dunion* :: $'a \text{ set} \Rightarrow 'a \text{ set set} \Rightarrow 'a \text{ set set}$

for $sp :: 'a \text{ set}$ **and** $A :: 'a \text{ set set}$
where
 $Basic\text{-}cd[intro, simp]: a \in A \implies a \in \text{sigma-sets-cinter-dunion } sp \ A$
 $| Top\text{-}cd[simp]: sp \in \text{sigma-sets-cinter-dunion } sp \ A$
 $| Inter\text{-}cd: (\bigwedge i::nat. a \ i \in \text{sigma-sets-cinter-dunion } sp \ A) \implies (\bigcap i. a \ i) \in \text{sigma-sets-cinter-dunion } sp \ A$
 $| Union\text{-}cd: (\bigwedge i::nat. a \ i \in \text{sigma-sets-cinter-dunion } sp \ A) \implies \text{disjoint-family } a \implies (\bigcup i. a \ i) \in \text{sigma-sets-cinter-dunion } sp \ A$

lemma $\text{sigma-sets-cinter-dunion-subset}: \text{sigma-sets-cinter-dunion } sp \ A \subseteq \text{sigma-sets-cinter } sp \ A$
 $\langle proof \rangle$

lemma $\text{sigma-sets-cinter-into-sp}$:
 assumes $A \subseteq Pow \ sp \ x \in \text{sigma-sets-cinter } sp \ A$
 shows $x \subseteq sp$
 $\langle proof \rangle$

lemma $\text{sigma-sets-cinter-dunion-into-sp}$:
 assumes $A \subseteq Pow \ sp \ x \in \text{sigma-sets-cinter-dunion } sp \ A$
 shows $x \subseteq sp$
 $\langle proof \rangle$

lemma $\text{sigma-sets-cinter-int}$:
 assumes $a \in \text{sigma-sets-cinter } sp \ A \ b \in \text{sigma-sets-cinter } sp \ A$
 shows $a \cap b \in \text{sigma-sets-cinter } sp \ A$
 $\langle proof \rangle$

lemma $\text{sigma-sets-cinter-dunion-int}$:
 assumes $a \in \text{sigma-sets-cinter-dunion } sp \ A \ b \in \text{sigma-sets-cinter-dunion } sp \ A$
 shows $a \cap b \in \text{sigma-sets-cinter-dunion } sp \ A$
 $\langle proof \rangle$

lemma $\text{sigma-sets-cinter-un}$:
 assumes $a \in \text{sigma-sets-cinter } sp \ A \ b \in \text{sigma-sets-cinter } sp \ A$
 shows $a \cup b \in \text{sigma-sets-cinter } sp \ A$
 $\langle proof \rangle$

lemma $\text{sigma-sets-eq-cinter-dunion}$:
 assumes $\text{metrizable-space } X$
 shows $\text{sigma-sets } (topspace \ X) \ \{U. \text{openin } X \ U\} = \text{sigma-sets-cinter-dunion } (topspace \ X) \ \{U. \text{openin } X \ U\}$
 $\langle proof \rangle$

lemma $\text{sigma-sets-eq-cinter}$:
 assumes $\text{metrizable-space } X$
 shows $\text{sigma-sets } (topspace \ X) \ \{U. \text{openin } X \ U\} = \text{sigma-sets-cinter } (topspace \ X) \ \{U. \text{openin } X \ U\}$
 $\langle proof \rangle$

1.2.2 Measurable Isomorphisms

definition *measurable-isomorphic-map*::['a measure, 'b measure, 'a $\Rightarrow$ 'b] $\Rightarrow$ bool
where

measurable-isomorphic-map $M\ N\ f \longleftrightarrow \text{bij-betw } f\ (\text{space } M)\ (\text{space } N) \wedge f \in M \rightarrow_M N \wedge \text{the-inv-into } (\text{space } M)\ f \in N \rightarrow_M M$

lemma *measurable-isomorphic-map-sets-cong*:

assumes *sets* $M = \text{sets } M'\ \text{sets } N = \text{sets } N'$

shows *measurable-isomorphic-map* $M\ N\ f \longleftrightarrow \text{measurable-isomorphic-map } M'\ N'\ f$

<proof>

lemma *measurable-isomorphic-map-surj*:

assumes *measurable-isomorphic-map* $M\ N\ f$

shows $f\ ' \text{space } M = \text{space } N$

<proof>

lemma *measurable-isomorphic-mapI*:

assumes *bij-betw* $f\ (\text{space } M)\ (\text{space } N)\ f \in M \rightarrow_M N\ \text{the-inv-into } (\text{space } M)\ f \in N \rightarrow_M M$

shows *measurable-isomorphic-map* $M\ N\ f$

<proof>

lemma *measurable-isomorphic-map-byWitness*:

assumes $f \in M \rightarrow_M N\ g \in N \rightarrow_M M \wedge x. x \in \text{space } M \implies g\ (f\ x) = x \wedge x. x \in \text{space } N \implies f\ (g\ x) = x$

shows *measurable-isomorphic-map* $M\ N\ f$

<proof>

lemma *measurable-isomorphic-map-restrict-space*:

assumes $f \in M \rightarrow_M N \wedge A. A \in \text{sets } M \implies f\ ' A \in \text{sets } N\ \text{inj-on } f\ (\text{space } M)$

shows *measurable-isomorphic-map* $M\ (\text{restrict-space } N\ (f\ ' \text{space } M))\ f$

<proof>

lemma *measurable-isomorphic-mapD'*:

assumes *measurable-isomorphic-map* $M\ N\ f$

shows $\wedge A. A \in \text{sets } M \implies f\ ' A \in \text{sets } N\ f \in M \rightarrow_M N$

$\exists g. \text{bij-betw } g\ (\text{space } N)\ (\text{space } M) \wedge g \in N \rightarrow_M M \wedge (\forall x \in \text{space } M. g\ (f\ x) = x) \wedge (\forall x \in \text{space } N. f\ (g\ x) = x) \wedge (\forall A \in \text{sets } N. g\ ' A \in \text{sets } M)$

<proof>

lemma *measurable-isomorphic-map-inv*:

assumes *measurable-isomorphic-map* $M\ N\ f$

shows *measurable-isomorphic-map* $N\ M\ (\text{the-inv-into } (\text{space } M)\ f)$

<proof>

lemma *measurable-isomorphic-map-comp*:

assumes *measurable-isomorphic-map* $M\ N\ f$ **and** *measurable-isomorphic-map* $N\ L\ g$

shows *measurable-isomorphic-map* $M\ L\ (g \circ f)$
 $\langle proof \rangle$

definition *measurable-isomorphic*::['a measure, 'b measure] $\Rightarrow$ bool (**infixr** $\langle measurable'-isomorphic \rangle$ 50) **where**
 $M\ measurable-isomorphic\ N \iff (\exists f. measurable-isomorphic-map\ M\ N\ f)$

lemma *measurable-isomorphic-sets-cong*:
assumes $sets\ M = sets\ M'\ sets\ N = sets\ N'$
shows $M\ measurable-isomorphic\ N \iff M'\ measurable-isomorphic\ N'$
 $\langle proof \rangle$

lemma *measurable-isomorphicD*:
assumes $M\ measurable-isomorphic\ N$
shows $\exists f\ g. f \in M \rightarrow_M N \wedge g \in N \rightarrow_M M \wedge (\forall x \in space\ M. g\ (f\ x) = x) \wedge$
 $(\forall y \in space\ N. f\ (g\ y) = y) \wedge (\forall A \in sets\ M. f\ 'A \in sets\ N) \wedge (\forall A \in sets\ N. g\ 'A$
 $\in sets\ M)$
 $\langle proof \rangle$

lemma *measurable-isomorphic-cardinality-eq*:
assumes $M\ measurable-isomorphic\ N$
shows $space\ M \approx space\ N$
 $\langle proof \rangle$

lemma *measurable-isomorphic-count-spaces*: $count-space\ A\ measurable-isomorphic$
 $count-space\ B \iff A \approx B$
 $\langle proof \rangle$

lemma *measurable-isomorphic-byWitness*:
assumes $f \in M \rightarrow_M N \bigwedge x. x \in space\ M \implies g\ (f\ x) = x$
and $g \in N \rightarrow_M M \bigwedge y. y \in space\ N \implies f\ (g\ y) = y$
shows $M\ measurable-isomorphic\ N$
 $\langle proof \rangle$

lemma *measurable-isomorphic-refl*:
 $M\ measurable-isomorphic\ M$
 $\langle proof \rangle$

lemma *measurable-isomorphic-sym*:
assumes $M\ measurable-isomorphic\ N$
shows $N\ measurable-isomorphic\ M$
 $\langle proof \rangle$

lemma *measurable-isomorphic-trans*:
assumes $M\ measurable-isomorphic\ N$ **and** $N\ measurable-isomorphic\ L$
shows $M\ measurable-isomorphic\ L$
 $\langle proof \rangle$

lemma *measurable-isomorphic-empty*:

assumes $\text{space } M = \{\}$ $\text{space } N = \{\}$
shows $M \text{ measurable-isomorphic } N$
 $\langle \text{proof} \rangle$

lemma *measurable-isomorphic-empty1*:
assumes $\text{space } M = \{\}$ $M \text{ measurable-isomorphic } N$
shows $\text{space } N = \{\}$
 $\langle \text{proof} \rangle$

lemma *measurable-isomorphic-empty2*:
assumes $\text{space } N = \{\}$ $M \text{ measurable-isomorphic } N$
shows $\text{space } M = \{\}$
 $\langle \text{proof} \rangle$

lemma *measurable-lift-product*:
assumes $\bigwedge i. i \in I \implies f \ i \in (M \ i) \rightarrow_M (N \ i)$
shows $(\lambda x \ i. \text{if } i \in I \text{ then } f \ i \ (x \ i) \text{ else undefined}) \in (\prod_M i \in I. M \ i) \rightarrow_M (\prod_{i \in I. N \ i})$
 $\langle \text{proof} \rangle$

lemma *measurable-isomorphic-map-lift-product*:
assumes $\bigwedge i. i \in I \implies \text{measurable-isomorphic-map } (M \ i) \ (N \ i) \ (h \ i)$
shows $\text{measurable-isomorphic-map } (\prod_M i \in I. M \ i) \ (\prod_M i \in I. N \ i) \ (\lambda x \ i. \text{if } i \in I \text{ then } h \ i \ (x \ i) \text{ else undefined})$
 $\langle \text{proof} \rangle$

lemma *measurable-isomorphic-lift-product*:
assumes $\bigwedge i. i \in I \implies (M \ i) \text{ measurable-isomorphic } (N \ i)$
shows $(\prod_M i \in I. M \ i) \text{ measurable-isomorphic } (\prod_M i \in I. N \ i)$
 $\langle \text{proof} \rangle$

<https://math24.net/cantor-schroder-bernstein-theorem.html>

lemma *Schroeder-Bernstein-measurable'*:
assumes $f \ ' \ (\text{space } M) \in \text{sets } N \ g \ ' \ (\text{space } N) \in \text{sets } M$
and $\text{measurable-isomorphic-map } M \ (\text{restrict-space } N \ (f \ ' \ (\text{space } M))) \ f \text{ and }$
 $\text{measurable-isomorphic-map } N \ (\text{restrict-space } M \ (g \ ' \ (\text{space } N))) \ g$
shows $\exists h. \text{measurable-isomorphic-map } M \ N \ h$
 $\langle \text{proof} \rangle$

lemma *Schroeder-Bernstein-measurable*:
assumes $f \in M \rightarrow_M N \bigwedge A. A \in \text{sets } M \implies f \ ' \ A \in \text{sets } N \text{ inj-on } f \ (\text{space } M)$
and $g \in N \rightarrow_M M \bigwedge A. A \in \text{sets } N \implies g \ ' \ A \in \text{sets } M \text{ inj-on } g \ (\text{space } N)$
shows $\exists h. \text{measurable-isomorphic-map } M \ N \ h$
 $\langle \text{proof} \rangle$

lemma *measurable-isomorphic-from-embeddings*:
assumes $M \text{ measurable-isomorphic } (\text{restrict-space } N \ B) \ N \text{ measurable-isomorphic } (\text{restrict-space } M \ A)$
and $A \in \text{sets } M \ B \in \text{sets } N$

shows M measurable-isomorphic N
 $\langle proof \rangle$

lemma measurable-isomorphic-antisym:
assumes B measurable-isomorphic (restrict-space C c) A measurable-isomorphic
(restrict-space B b)
and $c \in \text{sets } C$ $b \in \text{sets } B$ C measurable-isomorphic A
shows C measurable-isomorphic B
 $\langle proof \rangle$

lemma countable-infinite-isomorphisc-to-nat-index:
assumes countable I **and** infinite I
shows $(\Pi_M \ x \in I. \ M)$ measurable-isomorphic $(\Pi_M \ (x :: nat) \in UNIV. \ M)$
 $\langle proof \rangle$

lemma PiM-PiM-isomorphic-to-PiM:
 $(\Pi_M \ i \in I. \ \Pi_M \ j \in J. \ M \ i \ j)$ measurable-isomorphic $(\Pi_M \ (i, j) \in I \times J. \ M \ i \ j)$
 $\langle proof \rangle$

lemma measurable-isomorphic-map-sigma-sets:
assumes sets $M = \text{sigma-sets } (\text{space } M)$ U measurable-isomorphic-map $M \ N \ f$
shows sets $N = \text{sigma-sets } (\text{space } N)$ $((\cdot) \ f \cdot U)$
 $\langle proof \rangle$

1.2.3 Borel Spaces Genereted from Abstract Topologies

definition borel-of :: 'a topology $\Rightarrow$ 'a measure **where**
borel-of $X \equiv \text{sigma } (\text{topspace } X) \ \{U. \text{openin } X \ U\}$

lemma emeasure-borel-of: emeasure (borel-of X) $A = 0$
 $\langle proof \rangle$

lemma borel-of-euclidean: borel-of euclidean = borel
 $\langle proof \rangle$

lemma space-borel-of: space (borel-of X) = topspace X
 $\langle proof \rangle$

lemma sets-borel-of: sets (borel-of X) = sigma-sets (topspace X) $\{U. \text{openin } X \ U\}$
 $\langle proof \rangle$

lemma sets-borel-of-closed: sets (borel-of X) = sigma-sets (topspace X) $\{U. \text{closedin } X \ U\}$
 $\langle proof \rangle$

lemma borel-of-open:
assumes openin $X \ U$
shows $U \in \text{sets } (\text{borel-of } X)$

<proof>

lemma *borel-of-closed*:

assumes *closedin* $X\ U$

shows $U \in \text{sets } (\text{borel-of } X)$

<proof>

lemma(**in** *Metric-space*) *nbh-sets[measurable]*: $(\bigcup a \in A. \text{mball } a\ e) \in \text{sets } (\text{borel-of } \text{mtopology})$

<proof>

lemma *borel-of-gdelta-in*:

assumes *gdelta-in* $X\ U$

shows $U \in \text{sets } (\text{borel-of } X)$

<proof>

lemma *borel-of-subtopology*:

borel-of (subtopology $X\ U$) = *restrict-space (borel-of* X) U

<proof>

lemma *sets-borel-of-discrete-topology*: $\text{sets } (\text{borel-of } (\text{discrete-topology } I)) = \text{sets } (\text{count-space } I)$

<proof>

lemma *continuous-map-measurable*:

assumes *continuous-map* $X\ Y\ f$

shows $f \in \text{borel-of } X \rightarrow_M \text{borel-of } Y$

<proof>

lemma *upper-semicontinuous-map-measurable*:

fixes $f :: 'a \Rightarrow 'b :: \{\text{linorder-topology, second-countable-topology}\}$

assumes *upper-semicontinuous-map* $X\ f$

shows $f \in \text{borel-measurable } (\text{borel-of } X)$

<proof>

lemma *lower-semicontinuous-map-measurable*:

fixes $f :: 'a \Rightarrow 'b :: \{\text{linorder-topology, second-countable-topology}\}$

assumes *lower-semicontinuous-map* $X\ f$

shows $f \in \text{borel-measurable } (\text{borel-of } X)$

<proof>

lemma *open-map-preserves-sets*:

assumes *open-map* $S\ T\ f\ \text{inj-on } f\ (\text{topspace } S)\ A \in \text{sets } (\text{borel-of } S)$

shows $f\ 'A \in \text{sets } (\text{borel-of } T)$

<proof>

lemma *open-map-preserves-sets'*:

assumes *open-map* $S\ (\text{subtopology } T\ (f\ '(\text{topspace } S)))\ f\ \text{inj-on } f\ (\text{topspace } S)$
 $f\ '(\text{topspace } S) \in \text{sets } (\text{borel-of } T)\ A \in \text{sets } (\text{borel-of } S)$

shows $f \cdot A \in \text{sets } (\text{borel-of } T)$
 $\langle \text{proof} \rangle$

Abstract topology version of $\text{open} = \text{generate-topology } ?X \implies \text{borel} = \text{sigma } \text{UNIV } ?X$.

lemma *borel-of-second-countable'*:
assumes *second-countable* S **and** *subbase-in* $S \mathcal{U}$
shows $\text{borel-of } S = \text{sigma } (\text{topspace } S) \mathcal{U}$
 $\langle \text{proof} \rangle$

Abstract topology version $\text{borel} \otimes_M \text{borel} = \text{borel}$.

lemma *borel-of-prod*:
assumes *second-countable* S **and** *second-countable* S'
shows $\text{borel-of } S \otimes_M \text{borel-of } S' = \text{borel-of } (\text{prod-topology } S S')$
 $\langle \text{proof} \rangle$

lemma *product-borel-of-measurable*:
assumes $i \in I$
shows $(\lambda x. x \ i) \in (\text{borel-of } (\text{product-topology } S \ I)) \rightarrow_M \text{borel-of } (S \ i)$
 $\langle \text{proof} \rangle$

Abstract topology version of $\text{sets } (Pi_M \ \text{UNIV } (\lambda \cdot. \text{borel})) \subseteq \text{sets borel}$

lemma *sets-PiM-subset-borel-of*:
 $\text{sets } (\Pi_M \ i \in I. \text{borel-of } (S \ i)) \subseteq \text{sets } (\text{borel-of } (\text{product-topology } S \ I))$
 $\langle \text{proof} \rangle$

Abstract topology version of $\text{sets } (Pi_M \ \text{UNIV } (\lambda i. \text{borel})) = \text{sets borel}$.

lemma *sets-PiM-equal-borel-of*:
assumes *countable* I **and** $\bigwedge i. i \in I \implies \text{second-countable } (S \ i)$
shows $\text{sets } (\Pi_M \ i \in I. \text{borel-of } (S \ i)) = \text{sets } (\text{borel-of } (\text{product-topology } S \ I))$
 $\langle \text{proof} \rangle$

lemma *homeomorphic-map-borel-isomorphic*:
assumes *homeomorphic-map* $X \ Y \ f$
shows *measurable-isomorphic-map* $(\text{borel-of } X) \ (\text{borel-of } Y) \ f$
 $\langle \text{proof} \rangle$

lemma *homeomorphic-space-measurable-isomorphic*:
assumes S *homeomorphic-space* T
shows $\text{borel-of } S$ *measurable-isomorphic* $\text{borel-of } T$
 $\langle \text{proof} \rangle$

lemma *measurable-isomorphic-borel-map*:
assumes $\text{sets } M = \text{sets } (\text{borel-of } S)$ **and** $f: \text{measurable-isomorphic-map } M \ N \ f$
shows $\exists S'. \text{homeomorphic-map } S \ S' \ f \wedge \text{sets } N = \text{sets } (\text{borel-of } S')$
 $\langle \text{proof} \rangle$

lemma *measurable-isomorphic-borels*:

```

assumes sets  $M = \text{sets } (\text{borel-of } S) \text{ } M \text{ measurable-isomorphic } N$ 
shows  $\exists S'. S \text{ homeomorphic-space } S' \wedge \text{sets } N = \text{sets } (\text{borel-of } S')$ 
  <proof>

end

```

1.3 Lemmas for Abstract Metric Spaces

```

theory Set-Based-Metric-Space
imports Lemmas-StandardBorel
begin

```

We prove additional lemmas related to set-based metric spaces.

1.3.1 Basic Lemmas

```

lemma
assumes Metric-space  $M \text{ } d \wedge x \text{ } y. x \in M \implies y \in M \implies d \text{ } x \text{ } y = d' \text{ } x \text{ } y$ 
and  $\wedge x \text{ } y. d' \text{ } x \text{ } y = d' \text{ } y \text{ } x \wedge x \text{ } y. d' \text{ } x \text{ } y \geq 0$ 
shows Metric-space-eq: Metric-space  $M \text{ } d'$ 
and Metric-space-eq-mtopology: Metric-space.mtopology  $M \text{ } d = \text{Metric-space.mtopology } M \text{ } d'$ 
and Metric-space-eq-mcomplete: Metric-space.mcomplete  $M \text{ } d \longleftrightarrow \text{Metric-space.mcomplete } M \text{ } d'$ 
  <proof>

```

```

context Metric-space
begin

```

```

lemma mtopology-base-in-balls: base-in mtopology  $\{\text{mball } a \text{ } \varepsilon \mid a \in M \wedge \varepsilon > 0\}$ 
  <proof>

```

```

lemma closedin-metric2: closedin mtopology  $C \longleftrightarrow C \subseteq M \wedge (\forall x. x \in C \longleftrightarrow (\forall \varepsilon > 0. \text{mball } x \text{ } \varepsilon \cap C \neq \{\}))$ 
  <proof>

```

```

lemma openin-mtopology2:
  openin mtopology  $U \longleftrightarrow U \subseteq M \wedge (\forall x \text{ } x. \text{limitin mtopology } x \text{ } x \text{ sequentially} \wedge x \in U \longrightarrow (\exists N. \forall n \geq N. x \text{ } n \in U))$ 
  <proof>

```

```

lemma closure-of-mball: mtopology closure-of mball  $a \text{ } e \subseteq \text{mcball } a \text{ } e$ 
  <proof>

```

```

lemma interior-of-mcball: mball  $a \text{ } e \subseteq \text{mtopology interior-of mcball } a \text{ } e$ 
  <proof>

```

```

lemma isolated-points-of-mtopology:

```

mtopology isolated-points-of $A = \{x \in M \cap A. \forall xn. \text{range } xn \subseteq A \wedge \text{limitin mtopology } xn \text{ } x \text{ sequentially} \longrightarrow (\exists no. \forall n \geq no. xn \ n = x)\}$
 $\langle \text{proof} \rangle$

lemma *perfect-set-mball-infinite*:
assumes *perfect-set mtopology* A $a \in A$ $e > 0$
shows *infinite* ($\text{mball } a \ e$)
 $\langle \text{proof} \rangle$

lemma *MCauchy-dist-Cauchy*:
assumes *MCauchy* xn *MCauchy* yn
shows *Cauchy* $(\lambda n. d (xn \ n) (yn \ n))$
 $\langle \text{proof} \rangle$

1.3.2 Dense in Metric Spaces

abbreviation *mdense* $\equiv$ *dense-in mtopology*

<https://people.bath.ac.uk/mw2319/ma30252/sec-dense.html>

lemma *mdense-def*:
 $\text{mdense } U \longleftrightarrow U \subseteq M \wedge (\forall x \in M. \forall \varepsilon > 0. \text{mball } x \ \varepsilon \cap U \neq \{\})$
 $\langle \text{proof} \rangle$

corollary *mdense-balls-cover*:
assumes *mdense* U **and** $e > 0$
shows $(\bigcup u \in U. \text{mball } u \ e) = M$
 $\langle \text{proof} \rangle$

lemma *mdense-empty-iff*: $\text{mdense } \{\} \longleftrightarrow M = \{\}$
 $\langle \text{proof} \rangle$

lemma *mdense-M*: $\text{mdense } M$
 $\langle \text{proof} \rangle$

lemma *mdense-def2*:
 $\text{mdense } U \longleftrightarrow U \subseteq M \wedge (\forall x \in M. \forall \varepsilon > 0. \exists y \in U. d \ x \ y < \varepsilon)$
 $\langle \text{proof} \rangle$

lemma *mdense-def3*:
 $\text{mdense } U \longleftrightarrow U \subseteq M \wedge (\forall x \in M. \exists xn. \text{range } xn \subseteq U \wedge \text{limitin mtopology } xn \ x \text{ sequentially})$
 $\langle \text{proof} \rangle$

Diameter

definition *mdiameter* :: 'a set $\Rightarrow$ ennreal **where**
 $\text{mdiameter } A \equiv \bigcup \{ \text{ennreal } (d \ x \ y) \mid x \ y. x \in A \cap M \wedge y \in A \cap M \}$

lemma *mdiameter-empty[simp]*:
 $\text{mdiameter } \{\} = 0$

$\langle \text{proof} \rangle$

lemma *mdiameter-def2*:

assumes $A \subseteq M$

shows $\text{mdiameter } A = \bigsqcup \{ \text{ennreal } (d \ x \ y) \mid x \ y. \ x \in A \wedge y \in A \}$

$\langle \text{proof} \rangle$

lemma *mdiameter-subset*:

assumes $A \subseteq B$

shows $\text{mdiameter } A \leq \text{mdiameter } B$

$\langle \text{proof} \rangle$

lemma *mdiameter-cball-leq*: $\text{mdiameter } (\text{mcball } a \ \varepsilon) \leq \text{ennreal } (2 * \varepsilon)$

$\langle \text{proof} \rangle$

lemma *mdiameter-ball-leq*:

$\text{mdiameter } (\text{mball } a \ \varepsilon) \leq \text{ennreal } (2 * \varepsilon)$

$\langle \text{proof} \rangle$

lemma *mdiameter-is-sup*:

assumes $x \in A \cap M \ y \in A \cap M$

shows $d \ x \ y \leq \text{mdiameter } A$

$\langle \text{proof} \rangle$

lemma *mdiameter-is-sup'*:

assumes $x \in A \cap M \ y \in A \cap M \ \text{mdiameter } A \leq \text{ennreal } r \ r \geq 0$

shows $d \ x \ y \leq r$

$\langle \text{proof} \rangle$

lemma *mdiameter-le*:

assumes $\bigwedge x \ y. \ x \in A \implies y \in A \implies d \ x \ y \leq r$

shows $\text{mdiameter } A \leq r$

$\langle \text{proof} \rangle$

lemma *mdiameter-eq-closure*: $\text{mdiameter } (\text{mtopology closure-of } A) = \text{mdiameter } A$

$\langle \text{proof} \rangle$

lemma *mbounded-finite-mdiameter*: $\text{mbounded } A \longleftrightarrow A \subseteq M \wedge \text{mdiameter } A < \infty$

$\langle \text{proof} \rangle$

Distance between a point and a set.

definition *d-set* :: $'a \text{ set} \Rightarrow 'a \Rightarrow \text{real}$ **where**

$d\text{-set } A \equiv (\lambda x. \text{if } A \neq \{\} \wedge A \subseteq M \wedge x \in M \text{ then } \text{Inf } \{d \ x \ y \mid y. \ y \in A\} \text{ else } 0)$

lemma *d-set-nonneg[simp]*:

$d\text{-set } A \ x \geq 0$

$\langle \text{proof} \rangle$

lemma *d-set-bdd-below*[simp]:

bdd-below $\{d\ x\ y \mid y. y \in A\}$
 $\langle proof \rangle$

lemma *d-set-singleton*[simp]:

$x \in M \implies y \in M \implies d\text{-set}\ \{y\}\ x = d\ x\ y$
 $\langle proof \rangle$

lemma *d-set-empty*[simp]:

d-set $\{\}$ $x = 0$
 $\langle proof \rangle$

lemma *d-set-notin*:

$x \notin M \implies d\text{-set}\ A\ x = 0$
 $\langle proof \rangle$

lemma *d-set-inA*:

assumes $x \in A$
shows *d-set* $A\ x = 0$
 $\langle proof \rangle$

lemma *d-set-nzeroD*:

assumes *d-set* $A\ x \neq 0$
shows $A \subseteq M\ x \notin A\ A \neq \{\}$
 $\langle proof \rangle$

lemma *d-set-antimono*:

assumes $A \subseteq B\ A \neq \{\}\ B \subseteq M$
shows *d-set* $B\ x \leq d\text{-set}\ A\ x$
 $\langle proof \rangle$

lemma *d-set-bounded*:

assumes $\bigwedge y. y \in A \implies d\ x\ y < K\ K > 0$
shows *d-set* $A\ x < K$
 $\langle proof \rangle$

lemma *d-set-tr*:

assumes $x \in M\ y \in M$
shows *d-set* $A\ x \leq d\ x\ y + d\text{-set}\ A\ y$
 $\langle proof \rangle$

lemma *d-set-abs-le*:

assumes $x \in M\ y \in M$
shows $|d\text{-set}\ A\ x - d\text{-set}\ A\ y| \leq d\ x\ y$
 $\langle proof \rangle$

lemma *d-set-inA-le*:

assumes $y \in A$

shows $d\text{-set } A \ x \leq d \ x \ y$
 $\langle \text{proof} \rangle$

lemma $d\text{-set-ball-empty}$:
assumes $A \neq \{\}$ $A \subseteq M$ $e > 0$ $x \in M$ $m\text{ball } x \ e \cap A = \{\}$
shows $d\text{-set } A \ x \geq e$
 $\langle \text{proof} \rangle$

lemma $d\text{-set-closed-pos}$:
assumes $\text{closedin } m\text{topology } A$ $A \neq \{\}$ $x \in M$ $x \notin A$
shows $d\text{-set } A \ x > 0$
 $\langle \text{proof} \rangle$

lemma $g\text{delta-in-closed}$:
assumes $\text{closedin } m\text{topology } M$
shows $g\text{delta-in } m\text{topology } M$
 $\langle \text{proof} \rangle$

Oscillation

definition $\text{osc-on} :: ['b \text{ set}, 'b \text{ topology}, 'b \Rightarrow 'a, 'b] \Rightarrow \text{ennreal}$ **where**
 $\text{osc-on } A \ X \ f \equiv (\lambda y. \bigcap \{m\text{diameter } (f \text{ ` } (A \cap U)) \mid U. y \in U \wedge \text{openin } X \ U\})$

abbreviation $\text{osc } X \equiv \text{osc-on } (\text{topspace } X) \ X$

lemma osc-def : $\text{osc } X \ f = (\lambda y. \bigcap \{m\text{diameter } (f \text{ ` } U) \mid U. y \in U \wedge \text{openin } X \ U\})$
 $\langle \text{proof} \rangle$

lemma osc-on-less-iff :
 $\text{osc-on } A \ X \ f \ x < t \longleftrightarrow (\exists v. x \in v \wedge \text{openin } X \ v \wedge m\text{diameter } (f \text{ ` } (A \cap v)) < t)$
 $\langle \text{proof} \rangle$

lemma osc-less-iff :
 $\text{osc } X \ f \ x < t \longleftrightarrow (\exists v. x \in v \wedge \text{openin } X \ v \wedge m\text{diameter } (f \text{ ` } v) < t)$
 $\langle \text{proof} \rangle$

end

definition $\text{mdist-set} :: 'a \text{ metric} \Rightarrow 'a \text{ set} \Rightarrow 'a \Rightarrow \text{real}$ **where**
 $\text{mdist-set } m \equiv \text{Metric-space.d-set } (m\text{space } m) \ (\text{mdist } m)$

lemma(in Metric-space) mdist-set-Self : $\text{mdist-set } \text{Self} = d\text{-set}$
 $\langle \text{proof} \rangle$

lemma $\text{mdist-set-nonneg[simp]}$: $\text{mdist-set } m \ A \ x \geq 0$
 $\langle \text{proof} \rangle$

lemma $\text{mdist-set-singleton[simp]}$:
 $x \in m\text{space } m \implies y \in m\text{space } m \implies \text{mdist-set } m \ \{y\} \ x = \text{mdist } m \ x \ y$
 $\langle \text{proof} \rangle$

lemma *mdist-set-empty[simp]*: $\text{mdist-set } m \ \{\} \ x = 0$
 $\langle \text{proof} \rangle$

lemma *mdist-set-inA*:
assumes $x \in A$
shows $\text{mdist-set } m \ A \ x = 0$
 $\langle \text{proof} \rangle$

lemma *mdist-set-nzeroD*:
assumes $\text{mdist-set } m \ A \ x \neq 0$
shows $A \subseteq \text{mspace } m \ x \notin A \ A \neq \{\}$
 $\langle \text{proof} \rangle$

lemma *mdist-set-antimono*:
assumes $A \subseteq B \ A \neq \{\} \ B \subseteq \text{mspace } m$
shows $\text{mdist-set } m \ B \ x \leq \text{mdist-set } m \ A \ x$
 $\langle \text{proof} \rangle$

lemma *mdist-set-bounded*:
assumes $\bigwedge y. y \in A \implies \text{mdist } m \ x \ y < K \ K > 0$
shows $\text{mdist-set } m \ A \ x < K$
 $\langle \text{proof} \rangle$

lemma *mdist-set-tr*:
assumes $x \in \text{mspace } m \ y \in \text{mspace } m$
shows $\text{mdist-set } m \ A \ x \leq \text{mdist } m \ x \ y + \text{mdist-set } m \ A \ y$
 $\langle \text{proof} \rangle$

lemma *mdist-set-abs-le*:
assumes $x \in \text{mspace } m \ y \in \text{mspace } m$
shows $|\text{mdist-set } m \ A \ x - \text{mdist-set } m \ A \ y| \leq \text{mdist } m \ x \ y$
 $\langle \text{proof} \rangle$

lemma *mdist-set-inA-le*:
assumes $y \in A$
shows $\text{mdist-set } m \ A \ x \leq \text{mdist } m \ x \ y$
 $\langle \text{proof} \rangle$

lemma *mdist-set-ball-empty*:
assumes $A \neq \{\} \ A \subseteq \text{mspace } m \ e > 0 \ x \in \text{mspace } m \ \text{mball-of } m \ x \ e \cap A = \{\}$
shows $\text{mdist-set } m \ A \ x \geq e$
 $\langle \text{proof} \rangle$

lemma *mdist-set-closed-pos*:
assumes $\text{closedin } (m\text{topology-of } m) \ A \ A \neq \{\} \ x \in \text{mspace } m \ x \notin A$
shows $\text{mdist-set } m \ A \ x > 0$
 $\langle \text{proof} \rangle$

lemma *mdist-set-uniformly-continuous: uniformly-continuous-map m euclidean-metric*
(mdist-set m A)

<proof>

lemma *uniformly-continuous-map-add:*

fixes $f :: 'a \Rightarrow 'b :: \text{real-normed-vector}$

assumes *uniformly-continuous-map m euclidean-metric f uniformly-continuous-map*
m euclidean-metric g

shows *uniformly-continuous-map m euclidean-metric* $(\lambda x. f\ x + g\ x)$

<proof>

lemma *uniformly-continuous-map-real-divide:*

fixes $f :: 'a \Rightarrow \text{real}$

assumes *uniformly-continuous-map m euclidean-metric f uniformly-continuous-map*
m euclidean-metric g

and $\bigwedge x. x \in \text{mspace } m \implies g\ x \neq 0 \ \bigwedge x. x \in \text{mspace } m \implies |g\ x| \geq a \ a > 0$

$\bigwedge x. x \in \text{mspace } m \implies |g\ x| < Kg$

and $\bigwedge x. x \in \text{mspace } m \implies |f\ x| < Kf$

shows *uniformly-continuous-map m euclidean-metric* $(\lambda x. f\ x / g\ x)$

<proof>

lemma

assumes $e > 0$

shows *uniformly-continuous-map-from-capped-metric: uniformly-continuous-map*
(capped-metric e m1) m2 f $\longleftrightarrow$ uniformly-continuous-map m1 m2 f (is ?g1)

and *uniformly-continuous-map-to-capped-metric: uniformly-continuous-map m1*
(capped-metric e m2) f $\longleftrightarrow$ uniformly-continuous-map m1 m2 f (is ?g2)

<proof>

lemma *Urysohn-lemma-uniform:*

assumes *closedin (mtopology-of m) T closedin (mtopology-of m) U* $T \cap U = \{\}$
 $\bigwedge x\ y. x \in T \implies y \in U \implies \text{mdist } m\ x\ y \geq e \ e > 0$

obtains $f :: 'a \Rightarrow \text{real}$

where *uniformly-continuous-map m euclidean-metric f*

$\bigwedge x. f\ x \geq 0 \ \bigwedge x. f\ x \leq 1 \ \bigwedge x. x \in T \implies f\ x = 1 \ \bigwedge x. x \in U \implies f\ x = 0$

<proof>

Open maps

lemma *Metric-space-open-map-from-dist:*

assumes $f \in \text{mspace } m1 \rightarrow \text{mspace } m2$

and $\bigwedge x\ \varepsilon. x \in \text{mspace } m1 \implies \varepsilon > 0 \implies \exists \delta > 0. \forall y \in \text{mspace } m1. \text{mdist } m2$
 $(f\ x)\ (f\ y) < \delta \implies \text{mdist } m1\ x\ y < \varepsilon$

shows *open-map (mtopology-of m1) (subtopology (mtopology-of m2) (f ' mspace*
m1)) f

<proof>

1.3.3 Separability in Metric Spaces

context *Metric-space*

begin

For a metric space M , M is separable iff M is second countable.

lemma *generated-by-countable-balls:*

assumes *countable* U **and** *mdense* U

shows $mtopology = topology-generated-by \{mball\ y\ (1 / real\ n) \mid y\ n.\ y \in U\}$

<proof>

lemma *separable-space-imp-second-countable:*

assumes *separable-space* $mtopology$

shows *second-countable* $mtopology$

<proof>

corollary *separable-space-iff-second-countable:*

separable-space $mtopology \longleftrightarrow$ *second-countable* $mtopology$

<proof>

lemma *Lindelof-mdiameter:*

assumes *separable-space* $mtopology$ $0 < e$

shows $\exists U. countable\ U \wedge \bigcup U = M \wedge (\forall u \in U. mdiameter\ u < ennreal\ e)$

<proof>

end

lemma *metrizable-space-separable-iff-second-countable:*

assumes *metrizable-space* X

shows *separable-space* $X \longleftrightarrow$ *second-countable* X

<proof>

abbreviation *mdense-of* $m\ U \equiv dense-in\ (mtopology-of\ m)\ U$

lemma *mdense-of-def:* $mdense-of\ m\ U \longleftrightarrow (U \subseteq mspace\ m \wedge (\forall x \in mspace\ m. \forall \varepsilon > 0. mball-of\ m\ x\ \varepsilon \cap U \neq \{\}))$

<proof>

lemma *mdense-of-def2:* $mdense-of\ m\ U \longleftrightarrow (U \subseteq mspace\ m \wedge (\forall x \in mspace\ m. \forall \varepsilon > 0. \exists y \in U. mdist\ m\ x\ y < \varepsilon))$

<proof>

lemma *mdense-of-def3:* $mdense-of\ m\ U \longleftrightarrow (U \subseteq mspace\ m \wedge (\forall x \in mspace\ m. \exists xn. range\ xn \subseteq U \wedge limitin\ (mtopology-of\ m)\ xn\ x\ sequentially))$

<proof>

1.3.4 Compact Metric Spaces

context *Metric-space*

begin

lemma *mtotally-bounded-eq-compact-closedin:*

```

assumes mcomplete closedin mtopology S
shows mtotally-bounded S  $\longleftrightarrow$  S  $\subseteq$  M  $\wedge$  compactin mtopology S
 $\langle$ proof $\rangle$ 

lemma mtotally-bounded-def2: mtotally-bounded S  $\longleftrightarrow$  ( $\forall \varepsilon > 0. \exists K. \text{finite } K \wedge K$ 
 $\subseteq M \wedge S \subseteq (\bigcup_{x \in K} \text{mball } x \ \varepsilon))$ 
 $\langle$ proof $\rangle$ 

lemma compact-space-imp-separable:
assumes compact-space mtopology
shows separable-space mtopology
 $\langle$ proof $\rangle$ 

lemma separable-space-cfunspace:
assumes separable-space mtopology mcomplete
and metrizable-space X compact-space X
shows separable-space (mtopology-of (cfunspace X Self))
 $\langle$ proof $\rangle$ 

end

context Submetric
begin

lemma separable-sub:
assumes separable-space mtopology
shows separable-space sub.mtopology
 $\langle$ proof $\rangle$ 

end

```

1.3.5 Discrete Distance

```

lemma(in discrete-metric) separable-space-iff: separable-space disc.mtopology  $\longleftrightarrow$ 
countable M
 $\langle$ proof $\rangle$ 

```

1.3.6 Binary Product Metric Spaces

We define the L^1 -distance. L^1 -distance and L^2 distance (Euclid distance) generate the same topological space.

definition *prod-dist-L1 $\equiv \lambda d1 \ d2 \ (x,y) \ (x',y'). \ d1 \ x \ x' + d2 \ y \ y'$*

```

context Metric-space12
begin

```

```

lemma prod-L1-metric: Metric-space (M1  $\times$  M2) (prod-dist-L1 d1 d2)
 $\langle$ proof $\rangle$ 

```

sublocale *Prod-metric-L1*: *Metric-space* $M1 \times M2$ *prod-dist-L1* $d1\ d2$
 ⟨*proof*⟩

lemma *prod-dist-L1-geq*:
shows $d1\ x\ y \leq \text{prod-dist-L1}\ d1\ d2\ (x,x')\ (y,y')$
 $d2\ x'\ y' \leq \text{prod-dist-L1}\ d1\ d2\ (x,x')\ (y,y')$
 ⟨*proof*⟩

lemma *prod-dist-L1-ball*:
assumes $(x,x') \in \text{Prod-metric-L1.mball}\ (a,a')\ \varepsilon$
shows $x \in M1.mball\ a\ \varepsilon$
and $x' \in M2.mball\ a'\ \varepsilon$
 ⟨*proof*⟩

lemma *prod-dist-L1-ball'*:
assumes $z \in \text{Prod-metric-L1.mball}\ a\ \varepsilon$
shows $\text{fst}\ z \in M1.mball\ (\text{fst}\ a)\ \varepsilon$
and $\text{snd}\ z \in M2.mball\ (\text{snd}\ a)\ \varepsilon$
 ⟨*proof*⟩

lemma *prod-dist-L1-ball1'*: $\text{Prod-metric-L1.mball}\ (a1,a2)\ (\min\ e1\ e2) \subseteq M1.mball\ a1\ e1 \times M2.mball\ a2\ e2$
 ⟨*proof*⟩

lemma *prod-dist-L1-ball1*:
assumes $b1 \in M1.mball\ a1\ e1\ b2 \in M2.mball\ a2\ e2$
shows $\exists\ e12 > 0. \text{Prod-metric-L1.mball}\ (b1,b2)\ e12 \subseteq M1.mball\ a1\ e1 \times M2.mball\ a2\ e2$
 ⟨*proof*⟩

lemma *prod-dist-L1-ball2'*:
 $M1.mball\ a1\ e1 \times M2.mball\ a2\ e2 \subseteq \text{Prod-metric-L1.mball}\ (a1,a2)\ (e1 + e2)$
 ⟨*proof*⟩

lemma *prod-dist-L1-ball2*:
assumes $(b1,b2) \in \text{Prod-metric-L1.mball}\ (a1,a2)\ e12$
shows $\exists\ e1 > 0. \exists\ e2 > 0. M1.mball\ b1\ e1 \times M2.mball\ b2\ e2 \subseteq \text{Prod-metric-L1.mball}\ (a1,a2)\ e12$
 ⟨*proof*⟩

lemma *prod-dist-L1-mtopology*:
 $\text{Prod-metric-L1.mtopology} = \text{prod-topology}\ M1.mtopology\ M2.mtopology$
 ⟨*proof*⟩

lemma *prod-dist-L1-limitin-iff*: $\text{limitin}\ \text{Prod-metric-L1.mtopology}\ z_n\ z\ \text{sequentially} \iff \text{limitin}\ M1.mtopology\ (\lambda n. \text{fst}\ (z_n\ n))\ (\text{fst}\ z)\ \text{sequentially} \wedge \text{limitin}\ M2.mtopology\ (\lambda n. \text{snd}\ (z_n\ n))\ (\text{snd}\ z)\ \text{sequentially}$
 ⟨*proof*⟩

lemma *prod-dist-L1-MCauchy-iff*: $Prod-metric-L1.MCauchy\ zn \longleftrightarrow M1.MCauchy$
 $(\lambda n. fst\ (zn\ n)) \wedge M2.MCauchy\ (\lambda n. snd\ (zn\ n))$
 $\langle proof \rangle$

end

1.3.7 Sum Metric Spaces

locale *Sum-metric* =
fixes $I :: 'i\ set$
and $Mi :: 'i \Rightarrow 'a\ set$
and $di :: 'i \Rightarrow 'a \Rightarrow 'a \Rightarrow real$
assumes *Mi-disj*: *disjoint-family-on* $Mi\ I$
and *d-nonneg*: $\bigwedge i\ x\ y. 0 \leq di\ i\ x\ y$
and *d-bounded*: $\bigwedge i\ x\ y. di\ i\ x\ y < 1$
and *Md-metric*: $\bigwedge i. i \in I \implies Metric-space\ (Mi\ i)\ (di\ i)$
begin

abbreviation $M \equiv \bigcup_{i \in I}. Mi\ i$

lemma *Mi-inj-on*:
assumes $i \in I\ j \in I\ a \in Mi\ i\ a \in Mi\ j$
shows $i = j$
 $\langle proof \rangle$

definition *sum-dist* :: $['a, 'a] \Rightarrow real$ **where**
 $sum-dist\ x\ y \equiv (if\ x \in M \wedge y \in M\ then\ (if\ \exists i \in I. x \in Mi\ i \wedge y \in Mi\ i\ then\ di$
 $(THE\ i. i \in I \wedge x \in Mi\ i \wedge y \in Mi\ i)\ x\ y\ else\ 1)\ else\ 0)$

lemma *sum-dist-simps*:
shows $\bigwedge i. [i \in I; x \in Mi\ i; y \in Mi\ i] \implies sum-dist\ x\ y = di\ i\ x\ y$
and $\bigwedge i\ j. [i \in I; j \in I; i \neq j; x \in Mi\ i; y \in Mi\ j] \implies sum-dist\ x\ y = 1$
and $\bigwedge i. [i \in I; y \in M; x \in Mi\ i; y \notin Mi\ i] \implies sum-dist\ x\ y = 1$
and $\bigwedge i. [i \in I; x \in M; y \in Mi\ i; x \notin Mi\ i] \implies sum-dist\ x\ y = 1$
and $x \notin M \implies sum-dist\ x\ y = 0\ y \notin M \implies sum-dist\ x\ y = 0$
 $\langle proof \rangle$

lemma *sum-dist-if-less1*:
assumes $i \in I\ x \in Mi\ i\ y \in M\ sum-dist\ x\ y < 1$
shows $y \in Mi\ i$
 $\langle proof \rangle$

lemma *inM-cases*:
assumes $x \in M\ y \in M$
and $\bigwedge i. [i \in I; x \in Mi\ i; y \in Mi\ i] \implies P\ x\ y$
and $\bigwedge i\ j. [i \in I; j \in I; i \neq j; x \in Mi\ i; y \in Mi\ j; x \neq y] \implies P\ x\ y$
shows $P\ x\ y\ \langle proof \rangle$

sublocale *Sum-metric*: *Metric-space* $M\ sum-dist$

<proof>

lemma *sum-dist-le1*: *sum-dist x y ≤ 1*
<proof>

lemma *sum-dist-ball-eq-ball*:
 assumes *i ∈ I e ≤ 1 x ∈ Mi i*
 shows *Metric-space.mball (Mi i) (di i) x e = Sum-metric.mball x e*
<proof>

lemma *ball-le-sum-dist-ball*:
 assumes *i ∈ I*
 shows *Metric-space.mball (Mi i) (di i) x e ⊆ Sum-metric.mball x e*
<proof>

lemma *openin-mtopology-iff*:
 openin Sum-metric.mtopology U ⟷ U ⊆ M ∧ (∀ i ∈ I. openin (Metric-space.mtopology (Mi i) (di i)) (U ∩ Mi i))
<proof>

corollary *openin-mtopology-Mi*:
 assumes *i ∈ I*
 shows *openin Sum-metric.mtopology (Mi i)*
<proof>

corollary *subtopology-mtopology-Mi*:
 assumes *i ∈ I*
 shows *subtopology Sum-metric.mtopology (Mi i) = Metric-space.mtopology (Mi i) (di i)*
<proof>

lemma *limitin-Mi-limitin-M*:
 assumes *i ∈ I limitin (Metric-space.mtopology (Mi i) (di i)) xn x sequentially*
 shows *limitin Sum-metric.mtopology xn x sequentially*
<proof>

lemma *limitin-M-limitin-Mi*:
 assumes *limitin Sum-metric.mtopology xn x sequentially*
 shows *∃ i ∈ I. limitin (Metric-space.mtopology (Mi i) (di i)) xn x sequentially*
<proof>

lemma *MCauchy-Mi-MCauchy-M*:
 assumes *i ∈ I Metric-space.MCauchy (Mi i) (di i) xn*
 shows *Sum-metric.MCauchy xn*
<proof>

lemma *MCauchy-M-MCauchy-Mi*:
 assumes *Sum-metric.MCauchy xn*

shows $\exists m. \exists i \in I. \text{Metric-space.MCauchy } (Mi\ i) (di\ i) (\lambda n. xn\ (n + m))$
 $\langle proof \rangle$

lemma *separable-Mi-separable-M*:

assumes *countable I* $\bigwedge i. i \in I \implies \text{separable-space } (\text{Metric-space.mtopology } (Mi\ i) (di\ i))$

shows *separable-space Sum-metric.mtopology*
 $\langle proof \rangle$

lemma *separable-M-separable-Mi*:

assumes *separable-space Sum-metric.mtopology* $\bigwedge i. i \in I$

shows *separable-space* $(\text{Metric-space.mtopology } (Mi\ i) (di\ i))$
 $\langle proof \rangle$

lemma *mcomplete-Mi-mcomplete-M*:

assumes $\bigwedge i. i \in I \implies \text{Metric-space.mcomplete } (Mi\ i) (di\ i)$

shows *Sum-metric.mcomplete*
 $\langle proof \rangle$

lemma *mcomplete-M-mcomplete-Mi*:

assumes *Sum-metric.mcomplete* $i \in I$

shows *Metric-space.mcomplete* $(Mi\ i) (di\ i)$
 $\langle proof \rangle$

end

lemma *sum-metricI*:

fixes *Si*

assumes *disjoint-family-on Si I*

and $\bigwedge i\ x\ y. i \notin I \implies 0 \leq di\ i\ x\ y$

and $\bigwedge i\ x\ y. di\ i\ x\ y < 1$

and $\bigwedge i. i \in I \implies \text{Metric-space } (Si\ i) (di\ i)$

shows *Sum-metric I Si di*

$\langle proof \rangle$

end

1.3.8 Product Metric Spaces

theory *Set-Based-Metric-Product*

imports *Set-Based-Metric-Space*

begin

lemma *nsum-of-r'*:

fixes $r :: \text{real}$

assumes $r:0 < r < 1$

shows $(\sum n. r^n (n + k) * K) = r^k / (1 - r) * K$

(is ?lhs = -)
 <proof>

lemma *nsum-of-r-leq*:

fixes $r :: \text{real}$ **and** $a :: \text{nat} \Rightarrow \text{real}$
assumes $r:0 < r < 1$
and $a:\bigwedge n. 0 \leq a\ n \wedge n. a\ n \leq K$
shows $0 \leq (\sum n. r^{\wedge}(n+k) * a\ (n+l)) (\sum n. r^{\wedge}(n+k) * a\ (n+l)) \leq r^{\wedge}k$
 $/ (1 - r) * K$
 <proof>

lemma *nsum-of-r-le*:

fixes $r :: \text{real}$ **and** $a :: \text{nat} \Rightarrow \text{real}$
assumes $r:0 < r < 1$
and $a:\bigwedge n. 0 \leq a\ n \wedge n. a\ n \leq K \exists n' \geq l. a\ n' < K$
shows $(\sum n. r^{\wedge}(n+k) * a\ (n+l)) < r^{\wedge}k / (1 - r) * K$
 <proof>

definition *product-dist'* :: $[\text{real}, 'i \text{ set}, \text{nat} \Rightarrow 'i, 'i \Rightarrow 'a \text{ set}, 'i \Rightarrow 'a \Rightarrow 'a \Rightarrow \text{real}]$
 $\Rightarrow ('i \Rightarrow 'a) \Rightarrow ('i \Rightarrow 'a) \Rightarrow \text{real}$ **where**
product-dist-def: *product-dist'* $r\ I\ g\ Mi\ di \equiv (\lambda x\ y. \text{if } x \in (\prod_E i \in I. Mi\ i) \wedge y \in (\prod_E i \in I. Mi\ i) \text{ then } (\sum n. \text{if } g\ n \in I \text{ then } r^{\wedge}n * di\ (g\ n)\ (x\ (g\ n))\ (y\ (g\ n)) \text{ else } 0) \text{ else } 0)$

$$d(x, y) = \sum_{n \in \mathbb{N}} r^n * d_{g_I(i)}(x_{g_I(i)}, y_{g_I(i)}).$$

locale *Product-metric* =

fixes $r :: \text{real}$
and $I :: 'i \text{ set}$
and $f :: 'i \Rightarrow \text{nat}$
and $g :: \text{nat} \Rightarrow 'i$
and $Mi :: 'i \Rightarrow 'a \text{ set}$
and $di :: 'i \Rightarrow 'a \Rightarrow 'a \Rightarrow \text{real}$
and $K :: \text{real}$
assumes $r: 0 < r < 1$
and $I: \text{countable } I$
and *gf-comp-id* : $\bigwedge i. i \in I \implies g\ (f\ i) = i$
and *gf-if-finite*: $\text{finite } I \implies \text{bij-betw } f\ I\ \{.. < \text{card } I\}$
 $\text{finite } I \implies \text{bij-betw } g\ \{.. < \text{card } I\}\ I$
and *gf-if-infinite*: $\text{infinite } I \implies \text{bij-betw } f\ I\ \text{UNIV}$
 $\text{infinite } I \implies \text{bij-betw } g\ \text{UNIV}\ I$
 $\bigwedge n. \text{infinite } I \implies f\ (g\ n) = n$
and *Md-metric*: $\bigwedge i. i \in I \implies \text{Metric-space } (Mi\ i)\ (di\ i)$
and *di-nonneg*: $\bigwedge i\ x\ y. 0 \leq di\ i\ x\ y$
and *di-bounded*: $\bigwedge i\ x\ y. di\ i\ x\ y \leq K$
and *K-pos* : $0 < K$

lemma *from-nat-into-to-nat-on-product-metric-pair*:

assumes *countable* I
shows $\bigwedge i. i \in I \implies \text{from-nat-into } I\ (\text{to-nat-on } I\ i) = i$

and $\text{finite } I \implies \text{bij-betw } (\text{to-nat-on } I) \ I \ \{..< \text{card } I\}$
and $\text{finite } I \implies \text{bij-betw } (\text{from-nat-into } I) \ \{..< \text{card } I\} \ I$
and $\text{infinite } I \implies \text{bij-betw } (\text{to-nat-on } I) \ I \ \text{UNIV}$
and $\text{infinite } I \implies \text{bij-betw } (\text{from-nat-into } I) \ \text{UNIV } I$
and $\bigwedge n. \text{infinite } I \implies \text{to-nat-on } I \ (\text{from-nat-into } I \ n) = n$
 $\langle \text{proof} \rangle$

lemma *product-metric-pair-finite-nat*:
 $\text{bij-betw } \text{id } \{..n\} \ \{..< \text{card } \{..n\}\} \ \text{bij-betw } \text{id } \{..< \text{card } \{..n\}\} \ \{..n\}$
 $\langle \text{proof} \rangle$

lemma *product-metric-pair-finite-nat'*:
 $\text{bij-betw } \text{id } \{..<n\} \ \{..< \text{card } \{..<n\}\} \ \text{bij-betw } \text{id } \{..< \text{card } \{..<n\}\} \ \{..<n\}$
 $\langle \text{proof} \rangle$

context *Product-metric*
begin

abbreviation $\text{product-dist} \equiv \text{product-dist}' \ r \ I \ g \ Mi \ di$

lemma *nsum-of-rK*: $(\sum n. r \wedge (n + k) * K) = r \wedge k / (1 - r) * K$
 $\langle \text{proof} \rangle$

lemma *i-min*:
assumes $i \in I \ g \ n = i$
shows $f \ i \leq n$
 $\langle \text{proof} \rangle$

lemma *g-surj*:
assumes $i \in I$
shows $\exists n. \ g \ n = i$
 $\langle \text{proof} \rangle$

lemma *product-dist-summable'[simp]*:
 $\text{summable } (\lambda n. \ r \wedge n * di \ (g \ n) \ (x \ (g \ n)) \ (y \ (g \ n)))$
 $\langle \text{proof} \rangle$

lemma *product-dist-summable[simp]*:
 $\text{summable } (\lambda n. \ \text{if } g \ n \in I \ \text{then } r \wedge n * di \ (g \ n) \ (x \ (g \ n)) \ (y \ (g \ n)) \ \text{else } 0)$
 $\langle \text{proof} \rangle$

lemma *summable-rK[simp]*: $\text{summable } (\lambda n. \ r \wedge n * K)$
 $\langle \text{proof} \rangle$

lemma *Product-metric*: *Metric-space* $(\Pi_E \ i \in I. \ Mi \ i) \ \text{product-dist}$
 $\langle \text{proof} \rangle$

sublocale *Product-metric*: *Metric-space* $\Pi_E \ i \in I. \ Mi \ i \ \text{product-dist}$
 $\langle \text{proof} \rangle$

lemma *product-dist-leqr*: $\text{product-dist } x \ y \leq 1 / (1 - r) * K$
 <proof>

lemma *product-dist-geq*:
 assumes $i \in I$ and $g \ n = i \ x \in (\Pi_E \ i \in I. \text{Mi } i) \ y \in (\Pi_E \ i \in I. \text{Mi } i)$
 shows $\text{di } i \ (x \ i) \ (y \ i) \leq (1/r)^n * \text{product-dist } x \ y$
 (is ?lhs $\leq$?rhs)
 <proof>

lemma *limitin-M-iff-limitin-Mi*:
 shows $\text{limitin } \text{Product-metric.mtopology } x \ x \text{ sequentially} \longleftrightarrow (\exists N. \forall n \geq N. (\forall i \in I. \ x \ n \ i \in \text{Mi } i) \wedge (\forall i. i \notin I \longrightarrow x \ n \ i = \text{undefined})) \wedge (\forall i \in I. \text{limitin } (\text{Metric-space.mtopology } (\text{Mi } i) \ (\text{di } i)) \ (\lambda n. \ x \ n \ i) \ (x \ i) \text{ sequentially}) \wedge x \in (\Pi_E \ i \in I. \text{Mi } i)$
 <proof>

lemma *Product-metric-mtopology-eq*: $\text{product-topology } (\lambda i. \text{Metric-space.mtopology } (\text{Mi } i) \ (\text{di } i)) \ I = \text{Product-metric.mtopology}$
 <proof>

corollary *separable-Mi-separable-M*:
 assumes $\bigwedge i. i \in I \implies \text{separable-space } (\text{Metric-space.mtopology } (\text{Mi } i) \ (\text{di } i))$
 shows $\text{separable-space } \text{Product-metric.mtopology}$
 <proof>

lemma *mcomplete-Mi-mcomplete-M*:
 assumes $\bigwedge i. i \in I \implies \text{Metric-space.mcomplete } (\text{Mi } i) \ (\text{di } i)$
 shows $\text{Product-metric.mcomplete}$
 <proof>

end

lemma *product-metricI*:
 assumes $0 < r < 1$ countable $I \bigwedge i. i \in I \implies \text{Metric-space } (\text{Mi } i) \ (\text{di } i)$
 and $\bigwedge i \ x \ y. 0 \leq \text{di } i \ x \ y \bigwedge i \ x \ y. \text{di } i \ x \ y \leq K \ 0 < K$
 shows $\text{Product-metric } r \ I \ (\text{to-nat-on } I) \ (\text{from-nat-into } I) \ \text{Mi } \text{di } K$
 <proof>

lemma *product-metric-natI*:
 assumes $0 < r < 1 \bigwedge n. \text{Metric-space } (\text{Mi } n) \ (\text{di } n)$
 and $\bigwedge i \ x \ y. 0 \leq \text{di } i \ x \ y \bigwedge i \ x \ y. \text{di } i \ x \ y \leq K \ 0 < K$
 shows $\text{Product-metric } r \ \text{UNIV } \text{id } \text{id } \text{Mi } \text{di } K$
 <proof>

end

2 Abstract Polish Spaces

theory *Abstract-Metrizable-Topology*
 imports *Set-Based-Metric-Product*
begin

2.1 Polish Spaces

definition *Polish-space* $X \equiv \text{completely-metrizable-space } X \wedge \text{separable-space } X$

lemma(**in** *Metric-space*) *Polish-space-mtopology*:
 assumes *mcomplete separable-space mtopology*
 shows *Polish-space mtopology*
 $\langle \text{proof} \rangle$

lemma
 assumes *Polish-space X*
 shows *Polish-space-imp-completely-metrizable-space: completely-metrizable-space*
 X
 and *Polish-space-imp-metrizable-space: metrizable-space X*
 and *Polish-space-imp-second-countable: second-countable X*
 and *Polish-space-imp-separable-space: separable-space X*
 $\langle \text{proof} \rangle$

lemma *Polish-space-closedin*:
 assumes *Polish-space X closedin X A*
 shows *Polish-space (subtopology X A)*
 $\langle \text{proof} \rangle$

lemma *Polish-space-gdelta-in*:
 assumes *Polish-space X gdelta-in X A*
 shows *Polish-space (subtopology X A)*
 $\langle \text{proof} \rangle$

corollary *Polish-space-openin*:
 assumes *Polish-space X openin X A*
 shows *Polish-space (subtopology X A)*
 $\langle \text{proof} \rangle$

lemma *homeomorphic-Polish-space-aux*:
 assumes *Polish-space X X homeomorphic-space Y*
 shows *Polish-space Y*
 $\langle \text{proof} \rangle$

corollary *homeomorphic-Polish-space*:
 assumes *X homeomorphic-space Y*
 shows *Polish-space X $\longleftrightarrow$ Polish-space Y*
 $\langle \text{proof} \rangle$

lemma *Polish-space-euclidean[simp]*: *Polish-space (euclidean :: ('a :: polish-space))*

topology)
 ⟨proof⟩

lemma *Polish-space-countable[simp]*:
Polish-space (euclidean :: 'a :: {countable, discrete-topology} topology)
 ⟨proof⟩

lemma *Polish-space-discrete-topology*: *Polish-space (discrete-topology I) $\longleftrightarrow$ countable I*
 ⟨proof⟩

lemma *Polish-space-prod*:
 assumes *Polish-space X and Polish-space Y*
 shows *Polish-space (prod-topology X Y)*
 ⟨proof⟩

lemma *Polish-space-product*:
 assumes *countable I and $\bigwedge i. i \in I \implies \text{Polish-space } (S\ i)$*
 shows *Polish-space (product-topology S I)*
 ⟨proof⟩

lemma(in *Product-metric*) *Polish-spaceI*:
 assumes $\bigwedge i. i \in I \implies \text{separable-space } (\text{Metric-space.mtopology } (M\ i) (d\ i))$
 and $\bigwedge i. i \in I \implies \text{Metric-space.mcomplete } (M\ i) (d\ i)$
 shows *Polish-space Product-metric.mtopology*
 ⟨proof⟩

lemma(in *Sum-metric*) *Polish-spaceI*:
 assumes *countable I*
 and $\bigwedge i. i \in I \implies \text{separable-space } (\text{Metric-space.mtopology } (M\ i) (d\ i))$
 and $\bigwedge i. i \in I \implies \text{Metric-space.mcomplete } (M\ i) (d\ i)$
 shows *Polish-space Sum-metric.mtopology*
 ⟨proof⟩

lemma *compact-metrizable-imp-Polish-space*:
 assumes *metrizable-space X compact-space X*
 shows *Polish-space X*
 ⟨proof⟩

2.2 Extended Reals and Non-Negative Extended Reals

lemma *Polish-space-ereal*: *Polish-space (euclidean :: ereal topology)*
 ⟨proof⟩

corollary *Polish-space-ennreal*: *Polish-space (euclidean :: ennreal topology)*
 ⟨proof⟩

2.3 Continuous Embddings

abbreviation *Hilbert-cube-topology* :: *(nat $\Rightarrow$ real) topology where*

Hilbert-cube-topology $\equiv$ (*product-topology* ($\lambda n. \text{top-of-set } \{0..1\}$) *UNIV*)

lemma *topspace-Hilbert-cube: topspace Hilbert-cube-topology* = ($\Pi_E x \in \text{UNIV}. \{0..1\}$)
 ⟨proof⟩

lemma *Polish-space-Hilbert-cube: Polish-space Hilbert-cube-topology*
 ⟨proof⟩

abbreviation *Cantor-space-topology* :: (*nat* $\Rightarrow$ *real*) *topology* **where**
Cantor-space-topology $\equiv$ (*product-topology* ($\lambda n. \text{top-of-set } \{0,1\}$) *UNIV*)

lemma *topspace-Cantor-space:*
topspace Cantor-space-topology = ($\Pi_E x \in \text{UNIV}. \{0,1\}$)
 ⟨proof⟩

lemma *Polish-space-Cantor-space: Polish-space Cantor-space-topology*
 ⟨proof⟩

corollary *completely-metrizable-space-homeo-image-gdelta-in:*
assumes *completely-metrizable-space* *X* *completely-metrizable-space* *Y* *B* $\subseteq$ *topspace*
Y X homeomorphic-space subtopology Y B
shows *gdelta-in Y B*
 ⟨proof⟩

2.3.1 Embedding into Hilbert Cube

lemma *embedding-into-Hilbert-cube:*
assumes *metrizable-space X separable-space X*
shows $\exists A \subseteq \text{topspace Hilbert-cube-topology}. X \text{ homeomorphic-space (subtopology Hilbert-cube-topology } A)$
 ⟨proof⟩

corollary *embedding-into-Hilbert-cube-gdelta-in:*
assumes *Polish-space X*
shows $\exists A. \text{gdelta-in Hilbert-cube-topology } A \wedge X \text{ homeomorphic-space (subtopology Hilbert-cube-topology } A)$
 ⟨proof⟩

2.3.2 Embedding from Cantor Space

lemma *embedding-from-Cantor-space:*
assumes *Polish-space X uncountable (topspace X)*
shows $\exists A. \text{gdelta-in } X A \wedge \text{Cantor-space-topology homeomorphic-space (subtopology } X A)$
 ⟨proof⟩

2.4 Borel Spaces generated from Polish Spaces

lemma *closedin-clopen-topology:*
assumes *Polish-space X closedin X a*

shows $\exists X'. \text{Polish-space } X' \wedge (\forall u. \text{openin } X \ u \longrightarrow \text{openin } X' \ u) \wedge \text{topspace } X = \text{topspace } X' \wedge \text{sets } (\text{borel-of } X) = \text{sets } (\text{borel-of } X') \wedge \text{openin } X' \ a \wedge \text{closedin } X' \ a$
 $\langle \text{proof} \rangle$

lemma *Polish-space-union-Polish*:

fixes $X :: \text{nat} \Rightarrow 'a \text{ topology}$
assumes $\bigwedge n. \text{Polish-space } (X \ n) \wedge n. \text{topspace } (X \ n) = X t \wedge x \ y. x \in X t \implies y \in X t \implies x \neq y \implies \exists Ox \ Oy. (\forall n. \text{openin } (X \ n) \ Ox) \wedge (\forall n. \text{openin } (X \ n) \ Oy) \wedge x \in Ox \wedge y \in Oy \wedge \text{disjnt } Ox \ Oy$
defines $Xun \equiv \text{topology-generated-by } (\bigcup n. \{u. \text{openin } (X \ n) \ u\})$
shows *Polish-space* Xun
 $\langle \text{proof} \rangle$

lemma *sets-clopen-topology*:

assumes *Polish-space* $X \ a \in \text{sets } (\text{borel-of } X)$
shows $\exists X'. \text{Polish-space } X' \wedge (\forall u. \text{openin } X \ u \longrightarrow \text{openin } X' \ u) \wedge \text{topspace } X = \text{topspace } X' \wedge \text{sets } (\text{borel-of } X) = \text{sets } (\text{borel-of } X') \wedge \text{openin } X' \ a \wedge \text{closedin } X' \ a$
 $\langle \text{proof} \rangle$

end

3 Standard Borel Spaces

3.1 Standard Borel Spaces

theory *StandardBorel*

imports *Abstract-Metrizable-Topology*
begin

locale *standard-borel* =

fixes $M :: 'a \text{ measure}$
assumes *Polish-space*: $\exists S. \text{Polish-space } S \wedge \text{sets } M = \text{sets } (\text{borel-of } S)$
begin

lemma *singleton-sets*:

assumes $x \in \text{space } M$
shows $\{x\} \in \text{sets } M$
 $\langle \text{proof} \rangle$

corollary *countable-sets*:

assumes $A \subseteq \text{space } M$ *countable* A
shows $A \in \text{sets } M$
 $\langle \text{proof} \rangle$

lemma *standard-borel-restrict-space*:

assumes $A \in \text{sets } M$
shows *standard-borel* $(\text{restrict-space } M \ A)$

<proof>

end

locale *standard-borel-ne* = *standard-borel* +
 assumes *space-ne*: *space* *M* $\neq \{\}$
begin

lemma *standard-borel-ne-restrict-space*:
 assumes *A* $\in$ *sets* *M* *A* $\neq \{\}$
 shows *standard-borel-ne* (*restrict-space* *M* *A*)
 <proof>

lemma *standard-borel*: *standard-borel* *M*
 <proof>

end

lemma *standard-borel-sets*:
 assumes *standard-borel* *M* **and** *sets* *M* = *sets* *N*
 shows *standard-borel* *N*
 <proof>

lemma *standard-borel-ne-sets*:
 assumes *standard-borel-ne* *M* **and** *sets* *M* = *sets* *N*
 shows *standard-borel-ne* *N*
 <proof>

lemma *pair-standard-borel*:
 assumes *standard-borel* *M* *standard-borel* *N*
 shows *standard-borel* (*M* $\otimes_M$ *N*)
 <proof>

lemma *pair-standard-borel-ne*:
 assumes *standard-borel-ne* *M* *standard-borel-ne* *N*
 shows *standard-borel-ne* (*M* $\otimes_M$ *N*)
 <proof>

lemma *product-standard-borel*:
 assumes *countable* *I*
 and $\bigwedge i. i \in I \implies \text{standard-borel } (M\ i)$
 shows *standard-borel* ($\prod_M i \in I. M\ i$)
 <proof>

lemma *product-standard-borel-ne*:
 assumes *countable* *I*
 and $\bigwedge i. i \in I \implies \text{standard-borel-ne } (M\ i)$
 shows *standard-borel-ne* ($\prod_M i \in I. M\ i$)
 <proof>

lemma *closed-set-standard-borel*[simp]:
fixes $U :: 'a :: \text{topological-space set}$
assumes *Polish-space* (*euclidean* :: $'a \text{ topology}$) *closed* U
shows *standard-borel* (*restrict-space borel* U)
<proof>

lemma *closed-set-standard-borel-ne*[simp]:
fixes $U :: 'a :: \text{topological-space set}$
assumes *Polish-space* (*euclidean* :: $'a \text{ topology}$) *closed* U $U \neq \{\}$
shows *standard-borel-ne* (*restrict-space borel* U)
<proof>

lemma *open-set-standard-borel*[simp]:
fixes $U :: 'a :: \text{topological-space set}$
assumes *Polish-space* (*euclidean* :: $'a \text{ topology}$) *open* U
shows *standard-borel* (*restrict-space borel* U)
<proof>

lemma *open-set-standard-borel-ne*[simp]:
fixes $U :: 'a :: \text{topological-space set}$
assumes *Polish-space* (*euclidean* :: $'a \text{ topology}$) *open* U $U \neq \{\}$
shows *standard-borel-ne* (*restrict-space borel* U)
<proof>

lemma *standard-borel-ne-borel*[simp]: *standard-borel-ne* (*borel* :: $('a :: \text{polish-space})$
measure)
and *standard-borel-ne-lborel*[simp]: *standard-borel-ne lborel*
<proof>

lemma *count-space-standard'*[simp]:
assumes *countable* I
shows *standard-borel* (*count-space* I)
<proof>

lemma *count-space-standard-ne*[simp]: *standard-borel-ne* (*count-space* ($UNIV :: (-$
 $:: \text{countable}) \text{ set}$))
<proof>

corollary *measure-pmf-standard-borel-ne*[simp]: *standard-borel-ne* (*measure-pmf* (p
 $:: (- :: \text{countable}) \text{ pmf}$))
<proof>

corollary *measure-spmf-standard-borel-ne*[simp]: *standard-borel-ne* (*measure-spmf*
 $(p :: (- :: \text{countable}) \text{ spmf})$)
<proof>

corollary *countable-standard-ne*[simp]:
standard-borel-ne (*borel* :: $'a :: \{\text{countable}, \text{t2-space}\}$ *measure*)

<proof>

lemma(in *standard-borel*) *countable-discrete-space*:
 assumes *countable* (*space M*)
 shows *sets M = Pow* (*space M*)
<proof>

lemma(in *standard-borel*) *measurable-isomorphic-standard*:
 assumes *M measurable-isomorphic N*
 shows *standard-borel N*
<proof>

lemma(in *standard-borel-ne*) *measurable-isomorphic-standard-ne*:
 assumes *M measurable-isomorphic N*
 shows *standard-borel-ne N*
<proof>

lemma(in *standard-borel*) *standard-borel-embed-measure*:
 assumes *inj-on f* (*space M*)
 shows *standard-borel* (*embed-measure M f*)
<proof>

corollary(in *standard-borel-ne*) *standard-borel-ne-embed-measure*:
 assumes *inj-on f* (*space M*)
 shows *standard-borel-ne* (*embed-measure M f*)
<proof>

lemma
 shows *standard-ne-ereal: standard-borel-ne* (*borel :: ereal measure*)
 and *standard-ne-ennreal: standard-borel-ne* (*borel :: ennreal measure*)
<proof>

Cantor space $\mathcal{C}$

definition *Cantor-space* :: (*nat* $\Rightarrow$ *real*) *measure* **where**
Cantor-space $\equiv (\Pi_M i \in UNIV. \text{restrict-space borel } \{0,1\})$

lemma *Cantor-space-standard-ne: standard-borel-ne Cantor-space*
<proof>

lemma *Cantor-space-borel*:
 sets (*borel-of Cantor-space-topology*) = *sets Cantor-space*
 (**is** ?*lhs* = -)
<proof>

Hilbert cube $\mathcal{H}$

definition *Hilbert-cube* :: (*nat* $\Rightarrow$ *real*) *measure* **where**
Hilbert-cube $\equiv (\Pi_M i \in UNIV. \text{restrict-space borel } \{0..1\})$

lemma *Hilbert-cube-standard-ne: standard-borel-ne Hilbert-cube*

<proof>

lemma *Hilbert-cube-borel:*

sets (borel-of Hilbert-cube-topology) = sets Hilbert-cube (is ?lhs = -)

<proof>

3.2 Isomorphism between $\mathcal{C}$ and $\mathcal{H}$

lemma *Cantor-space-isomorphic-to-Hilbert-cube:*

Cantor-space measurable-isomorphic Hilbert-cube

<proof>

3.3 Final Results

lemma(*in standard-borel*) *embedding-into-Hilbert-cube:*

$\exists A \in \text{sets Hilbert-cube. } M \text{ measurable-isomorphic (restrict-space Hilbert-cube } A)$

<proof>

lemma(*in standard-borel*) *embedding-from-Cantor-space:*

assumes *uncountable (space M)*

shows *$\exists A \in \text{sets } M. \text{Cantor-space measurable-isomorphic (restrict-space } M \ A)$*

<proof>

corollary(*in standard-borel*) *uncountable-isomorphic-to-Hilbert-cube:*

assumes *uncountable (space M)*

shows *Hilbert-cube measurable-isomorphic M*

<proof>

corollary(*in standard-borel*) *uncountable-isomorphic-to-real:*

assumes *uncountable (space M)*

shows *$M \text{ measurable-isomorphic (borel :: real measure)}$*

<proof>

lemma(*in standard-borel*) *isomorphic-subset-real:*

assumes *$A \in \text{sets (borel :: real measure) uncountable } A$*

obtains *$B \text{ where } B \in \text{sets borel } B \subseteq A \ M \text{ measurable-isomorphic restrict-space borel } B$*

<proof>

lemma(*in standard-borel*) *countable-isomorphic-to-subset-real:*

assumes *countable (space M)*

obtains *$A :: \text{real set}$*

where *countable A A $\in \text{sets borel } M \text{ measurable-isomorphic restrict-space borel } A$*

<proof>

theorem *Borel-isomorphism-theorem:*

assumes *standard-borel M standard-borel N*

shows *$\text{space } M \approx \text{space } N \longleftrightarrow M \text{ measurable-isomorphic } N$*

<proof>

definition *to-real-on* :: 'a measure $\Rightarrow$ 'a $\Rightarrow$ real **where**
to-real-on $M \equiv$ (if uncountable (space M) then (SOME f . measurable-isomorphic-map
 M (borel :: real measure) f) else (real $\circ$ to-nat-on (space M)))

definition *from-real-into* :: 'a measure $\Rightarrow$ real $\Rightarrow$ 'a **where**
from-real-into $M \equiv$ (if uncountable (space M) then the-inv-into (space M) (to-real-on
 M) else (λr . from-nat-into (space M) (nat $\lfloor r \rfloor$)))

context *standard-borel*
begin

abbreviation *to-real* $\equiv$ *to-real-on* M
abbreviation *from-real* $\equiv$ *from-real-into* M

lemma *to-real-def-countable*:
assumes countable (space M)
shows *to-real* = (λr . real (to-nat-on (space M) r))
 $\langle$ proof $\rangle$

lemma *from-real-def-countable*:
assumes countable (space M)
shows *from-real* = (λr . from-nat-into (space M) (nat $\lfloor r \rfloor$))
 $\langle$ proof $\rangle$

lemma *from-real-to-real[simp]*:
assumes $x \in$ space M
shows *from-real* (to-real x) = x
 $\langle$ proof $\rangle$

lemma *to-real-measurable[measurable]*:
to-real $\in M \rightarrow_M$ borel
 $\langle$ proof $\rangle$

lemma *from-real-measurable'*:
assumes space $M \neq \{\}$
shows *from-real* $\in$ borel $\rightarrow_M M$
 $\langle$ proof $\rangle$

lemma *to-real-from-real*:
assumes uncountable (space M)
shows to-real (from-real r) = r
 $\langle$ proof $\rangle$

end

lemma(in *standard-borel-ne*) *from-real-measurable[measurable]*: *from-real* $\in$ borel
 $\rightarrow_M M$
 $\langle$ proof $\rangle$

end

References

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