Arrow's General Possibility Theorem

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Contents

1 Overview

This is a fairly literal encoding of some of Armatya Sen's proofs [Sen70] in Isabelle/HOL. The author initially wrote it while learning to use the proof assistant, and some locutions remain naive. This work is somewhat complementary to the mechanisation of more recent proofs of Arrow's Theorem and the Gibbard-Satterthwaite Theorem by Tobias Nipkow [Nip08].

I strongly recommend Sen's book to anyone interested in social choice theory; his proofs are quite lucid and accessible, and he situates the theory quite well within the broader economic tradition.

2 General Lemmas

2.1 Extra Finite-Set Lemmas

Small variant of *Finite-Set.finite-subset-induct*: also assume $F \subseteq A$ in the induction hypothesis.

lemma *finite-subset-induct′* [*consumes 2*, *case-names empty insert*]:
  assumes *finite F* **and** $F \subseteq A$
    and *empty*: P {}
    and *insert*: $\bigwedge a\ F.\ [\![\mathit{finite}\ F;\ a \in A;\ F \subseteq A;\ a \notin F;\ P\ F\ ]\!] \Longrightarrow P\ (\mathit{insert}\ a\ F)$
  shows $P\ F$
⟨*proof*⟩

A slight improvement on *List.finite-list* - add *distinct*.

lemma *finite-list*: $\mathit{finite}\ A \Longrightarrow \exists\, l.\ \mathit{set}\ l = A \wedge \mathit{distinct}\ l$
⟨*proof*⟩

2.2 Extra bijection lemmas

lemma *bij-betw-onto*: $\mathit{bij\text{-}betw}\ f\ A\ B \Longrightarrow f\ `\ A = B$ ⟨*proof*⟩

lemma *inj-on-UnI*: $[\![\ \mathit{inj\text{-}on}\ f\ A;\ \mathit{inj\text{-}on}\ f\ B;\ f\ `\ (A - B) \cap f\ `\ (B - A) = \{\}\ ]\!] \Longrightarrow \mathit{inj\text{-}on}\ f\ (A \cup B)$
  ⟨*proof*⟩

lemma *card-compose-bij*:
  assumes *bijf*: *bij-betw f A A*
  shows $\mathit{card}\ \{\ a \in A.\ P\ (f\ a)\ \} = \mathit{card}\ \{\ a \in A.\ P\ a\ \}$
⟨*proof*⟩

lemma *card-eq-bij*:
  assumes *cardAB*: $\mathit{card}\ A = \mathit{card}\ B$
    and *finiteA*: *finite A* **and** *finiteB*: *finite B*
  obtains *f* **where** *bij-betw f A B*
⟨*proof*⟩

lemma *bij-combine*:
  assumes *ABCD*: $A \subseteq B\ C \subseteq D$
    and *bijf*: *bij-betw f A C*
    and *bijg*: $\mathit{bij\text{-}betw}\ g\ (B - A)\ (D - C)$

obtains h
where $\mathit{bij\text{-}betw}\ h\ B\ D$
and $\bigwedge x.\ x \in A \Longrightarrow h\ x = f\ x$
and $\bigwedge x.\ x \in B - A \Longrightarrow h\ x = g\ x$
⟨*proof*⟩

lemma *bij-complete*:
assumes *finiteC*: $\mathit{finite}\ C$
and *ABC*: $A \subseteq C\ B \subseteq C$
and *bijf*: $\mathit{bij\text{-}betw}\ f\ A\ B$
obtains f' **where** $\mathit{bij\text{-}betw}\ f'\ C\ C$
and $\bigwedge x.\ x \in A \Longrightarrow f'\ x = f\ x$
and $\bigwedge x.\ x \in C - A \Longrightarrow f'\ x \in C - B$
⟨*proof*⟩

lemma *card-greater*:
assumes *finiteA*: $\mathit{finite}\ A$
and *c*: $\mathit{card}\ \{\ x \in A.\ P\ x\ \} > \mathit{card}\ \{\ x \in A.\ Q\ x\ \}$
obtains C
where $\mathit{card}\ (\{\ x \in A.\ P\ x\ \} - C) = \mathit{card}\ \{\ x \in A.\ Q\ x\ \}$
and $C \neq \{\}$
and $C \subseteq \{\ x \in A.\ P\ x\ \}$
⟨*proof*⟩

2.3 Collections of witnesses: *hasw*, *has*

Given a set of cardinality at least n, we can find up to n distinct witnesses. The built-in *card* function unfortunately satisfies:

$$\mathit{Finite\text{-}Set.card.infinite}\text{: } \mathit{infinite}\ A \Longrightarrow \mathit{card}\ A = \mathit{0}$$

These lemmas handle the infinite case uniformly.

Thanks to Gerwin Klein suggesting this approach.

definition $\mathit{hasw} :: {'a}\ \mathit{list} \Rightarrow {'a}\ \mathit{set} \Rightarrow \mathit{bool}$ **where**
$\mathit{hasw}\ \mathit{xs}\ S \equiv \mathit{set}\ \mathit{xs} \subseteq S \wedge \mathit{distinct}\ \mathit{xs}$

definition $\mathit{has} :: \mathit{nat} \Rightarrow {'a}\ \mathit{set} \Rightarrow \mathit{bool}$ **where**
$\mathit{has}\ n\ S \equiv \exists\,\mathit{xs}.\ \mathit{hasw}\ \mathit{xs}\ S \wedge \mathit{length}\ \mathit{xs} = n$

declare *hasw-def*[*simp*]

lemma *hasI*[*intro*]: $\mathit{hasw}\ \mathit{xs}\ S \Longrightarrow \mathit{has}\ (\mathit{length}\ \mathit{xs})\ S$ ⟨*proof*⟩

lemma *card-has*:
assumes *cardS*: $\mathit{card}\ S = n$
shows $\mathit{has}\ n\ S$
⟨*proof*⟩

lemma *card-has-rev*:
assumes *finiteS*: $\mathit{finite}\ S$
shows $\mathit{has}\ n\ S \Longrightarrow \mathit{card}\ S \geq n$ (**is** *?lhs* $\Longrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *has-0*: *has 0 S* ⟨*proof*⟩

lemma *has-suc-notempty*: *has* (*Suc n*) *S* $\Longrightarrow$ {} $\neq$ *S*
⟨*proof*⟩

lemma *has-suc-subset*: *has* (*Suc n*) *S* $\Longrightarrow$ {} $\subset$ *S*
⟨*proof*⟩

lemma *has-notempty-1*:
assumes *Sne*: *S* $\neq$ {}
shows *has 1 S*
⟨*proof*⟩

lemma *has-le-has*:
assumes *h*: *has n S*
and *nn'*: $n' \leq n$
shows *has n' S*
⟨*proof*⟩

lemma *has-ge-has-not*:
assumes *h*: ¬*has n S*
and *nn'*: $n \leq n'$
shows ¬*has n' S*
⟨*proof*⟩

lemma *has-eq*:
assumes *h*: *has n S*
and *hn'*: ¬*has* (*Suc n*) *S*
shows *card S* = *n*
⟨*proof*⟩

lemma *has-extend-witness*:
assumes *h*: *has n S*
shows ⟦ *set xs* $\subseteq$ *S*; *length xs* < *n* ⟧ $\Longrightarrow$ *set xs* $\subset$ *S*
⟨*proof*⟩

lemma *has-extend-witness'*:
⟦ *has n S*; *hasw xs S*; *length xs* < *n* ⟧ $\Longrightarrow$ ∃ *x*. *hasw* (*x* # *xs*) *S*
⟨*proof*⟩

lemma *has-witness-two*:
assumes *hasnS*: *has n S*
and *nn'*: $2 \leq n$
shows ∃ *x y*. *hasw* [*x*,*y*] *S*
⟨*proof*⟩

lemma *has-witness-three*:
assumes *hasnS*: *has n S*
and *nn'*: $3 \leq n$
shows ∃ *x y z*. *hasw* [*x*,*y*,*z*] *S*
⟨*proof*⟩

lemma *finite-set-singleton-contra*:
assumes *finiteS*: *finite S*
and *Sne*: $S \neq \{\}$
and *cardS*: *card* $S > 1 \Longrightarrow$ *False*
shows $\exists j.\ S = \{j\}$
⟨*proof*⟩

3 Preliminaries

The auxiliary concepts defined here are standard [Rou79, Sen70, Tay05]. Throughout we make use of a fixed set A of alternatives, drawn from some arbitrary type $'a$ of suitable size. Taylor [Tay05] terms this set an *agenda*. Similarly we have a type $'i$ of individuals and a population *Is*.

3.1 Rational Preference Relations (RPRs)

Definitions for rational preference relations (RPRs), which represent indifference or strict preference amongst some set of alternatives. These are also called *weak orders* or (ambiguously) *ballots*.

Unfortunately Isabelle's standard ordering operators and lemmas are typeclass-based, and as introducing new types is painful and we need several orders per type, we need to repeat some things.

type-synonym $'a$ *RPR* = $('a * 'a)$ *set*

abbreviation *rpr-eq-syntax* :: $'a \Rightarrow 'a$ *RPR* $\Rightarrow 'a \Rightarrow$ *bool* (‹- -⪯ -› [*50*, *1000*, *51*] *50*) **where**
$x\ {}_r{\preceq}\ y == (x, y) \in r$

definition *indifferent-pref* :: $'a \Rightarrow 'a$ *RPR* $\Rightarrow 'a \Rightarrow$ *bool* (‹- -≈ -› [*50*, *1000*, *51*] *50*) **where**
$x\ {}_r{\approx}\ y \equiv (x\ {}_r{\preceq}\ y \wedge y\ {}_r{\preceq}\ x)$

lemma *indifferent-prefI*[*intro*]: $[\![\ x\ {}_r{\preceq}\ y;\ y\ {}_r{\preceq}\ x\ ]\!] \Longrightarrow x\ {}_r{\approx}\ y$
⟨*proof*⟩

lemma *indifferent-prefD*[*dest*]: $x\ {}_r{\approx}\ y \Longrightarrow x\ {}_r{\preceq}\ y \wedge y\ {}_r{\preceq}\ x$
⟨*proof*⟩

definition *strict-pref* :: $'a \Rightarrow 'a$ *RPR* $\Rightarrow 'a \Rightarrow$ *bool* (‹- -≺ -› [*50*, *1000*, *51*] *50*) **where**
$x\ {}_r{\prec}\ y \equiv (x\ {}_r{\preceq}\ y \wedge \neg(y\ {}_r{\preceq}\ x))$

lemma *strict-pref-def-irrefl*[*simp*]: $\neg\ (x\ {}_r{\prec}\ x)$ ⟨*proof*⟩

lemma *strict-prefI*[*intro*]: $[\![\ x\ {}_r{\preceq}\ y;\ \neg(y\ {}_r{\preceq}\ x)\ ]\!] \Longrightarrow x\ {}_r{\prec}\ y$
⟨*proof*⟩

Traditionally, $x\ {}_r{\preceq}\ y$ would be written $x\ R\ y$, $x\ {}_r{\approx}\ y$ as $x\ I\ y$ and $x\ {}_r{\prec}\ y$ as $x\ P\ y$, where the relation r is implicit, and profiles are indexed by subscripting.

Complete means that every pair of distinct alternatives is ranked. The "distinct" part is a matter of taste, as it makes sense to regard an alternative as as good as itself. Here I take reflexivity separately.

definition *complete* :: *'a set* ⇒ *'a RPR* ⇒ *bool* **where**
complete A r ≡ $(\forall x \in A.\ \forall y \in A - \{x\}.\ x\ {}_r{\preceq}\ y \lor y\ {}_r{\preceq}\ x)$

lemma *completeI*[*intro*]:
$(\bigwedge x\ y.\ [\![\ x \in A;\ y \in A;\ x \neq y\ ]\!] \Longrightarrow x\ {}_r{\preceq}\ y \lor y\ {}_r{\preceq}\ x) \Longrightarrow$ *complete A r*
⟨*proof*⟩

lemma *completeD*[*dest*]:
$[\![$ *complete A r*; $x \in A;\ y \in A;\ x \neq y\ ]\!] \Longrightarrow x\ {}_r{\preceq}\ y \lor y\ {}_r{\preceq}\ x$
⟨*proof*⟩

lemma *complete-less-not*: $[\![$ *complete A r*; *hasw* $[x,y]$ A; $\neg\ x\ {}_r{\prec}\ y\ ]\!] \Longrightarrow y\ {}_r{\preceq}\ x$
⟨*proof*⟩

lemma *complete-indiff-not*: $[\![$ *complete A r*; *hasw* $[x,y]$ A; $\neg\ x\ {}_r{\approx}\ y\ ]\!] \Longrightarrow x\ {}_r{\prec}\ y \lor y\ {}_r{\prec}\ x$
⟨*proof*⟩

lemma *complete-exh*:
assumes *complete A r*
and *hasw* $[x,y]$ A
obtains (xPy) $x\ {}_r{\prec}\ y$
| (yPx) $y\ {}_r{\prec}\ x$
| (xIy) $x\ {}_r{\approx}\ y$
⟨*proof*⟩

Use the standard *refl*. Also define *irreflexivity* analogously to how *refl* is defined in the standard library.

declare *refl-onI*[*intro*] *refl-onD*[*dest*]

lemma *complete-refl-on*:
$[\![$ *complete A r*; *refl-on A r*; $x \in A;\ y \in A\ ]\!] \Longrightarrow x\ {}_r{\preceq}\ y \lor y\ {}_r{\preceq}\ x$
⟨*proof*⟩

definition *irrefl* :: *'a set* ⇒ *'a RPR* ⇒ *bool* **where**
irrefl A r ≡ $r \subseteq A \times A \land (\forall x \in A.\ \neg\ x\ {}_r{\preceq}\ x)$

lemma *irreflI*[*intro*]: $[\![\ r \subseteq A \times A;\ \bigwedge x.\ x \in A \Longrightarrow \neg\ x\ {}_r{\preceq}\ x\ ]\!] \Longrightarrow$ *irrefl A r*
⟨*proof*⟩

lemma *irreflD*[*dest*]: $[\![$ *irrefl A r*; $(x,\ y) \in r\ ]\!] \Longrightarrow$ *hasw* $[x,y]$ A
⟨*proof*⟩

lemma *irreflD'*[*dest*]:
$[\![$ *irrefl A r*; $r \neq \{\}\ ]\!] \Longrightarrow \exists x\ y.$ *hasw* $[x,y]$ $A \land (x,\ y) \in r$
⟨*proof*⟩

Rational preference relations, also known as weak orders and (I guess) complete pre-orders.

definition *rpr* :: *'a set* ⇒ *'a RPR* ⇒ *bool* **where**
rpr A r ≡ *complete A r* ∧ *refl-on A r* ∧ *trans r*

lemma *rprI*[*intro*]: $[\![$ *complete A r*; *refl-on A r*; *trans r* $]\!] \Longrightarrow$ *rpr A r*

⟨*proof*⟩

lemma *rprD*: $rpr\ A\ r \Longrightarrow complete\ A\ r \wedge refl\text{-}on\ A\ r \wedge trans\ r$
⟨*proof*⟩

lemma *rpr-in-set*[*dest*]: $\llbracket\ rpr\ A\ r;\ x\ {}_r\!\preceq\ y\ \rrbracket \Longrightarrow \{x,y\} \subseteq A$
⟨*proof*⟩

lemma *rpr-refl*[*dest*]: $\llbracket\ rpr\ A\ r;\ x \in A\ \rrbracket \Longrightarrow x\ {}_r\!\preceq\ x$
⟨*proof*⟩

lemma *rpr-less-not*: $\llbracket\ rpr\ A\ r;\ hasw\ [x,y]\ A;\ \neg\ x\ {}_r\!\prec\ y\ \rrbracket \Longrightarrow y\ {}_r\!\preceq\ x$
⟨*proof*⟩

lemma *rpr-less-imp-le*[*simp*]: $\llbracket\ x\ {}_r\!\prec\ y\ \rrbracket \Longrightarrow x\ {}_r\!\preceq\ y$
⟨*proof*⟩

lemma *rpr-less-imp-neq*[*simp*]: $\llbracket\ x\ {}_r\!\prec\ y\ \rrbracket \Longrightarrow x \neq y$
⟨*proof*⟩

lemma *rpr-less-trans*[*trans*]: $\llbracket\ x\ {}_r\!\prec\ y;\ y\ {}_r\!\prec\ z;\ rpr\ A\ r\ \rrbracket \Longrightarrow x\ {}_r\!\prec\ z$
⟨*proof*⟩

lemma *rpr-le-trans*[*trans*]: $\llbracket\ x\ {}_r\!\preceq\ y;\ y\ {}_r\!\preceq\ z;\ rpr\ A\ r\ \rrbracket \Longrightarrow x\ {}_r\!\preceq\ z$
⟨*proof*⟩

lemma *rpr-le-less-trans*[*trans*]: $\llbracket\ x\ {}_r\!\preceq\ y;\ y\ {}_r\!\prec\ z;\ rpr\ A\ r\ \rrbracket \Longrightarrow x\ {}_r\!\prec\ z$
⟨*proof*⟩

lemma *rpr-less-le-trans*[*trans*]: $\llbracket\ x\ {}_r\!\prec\ y;\ y\ {}_r\!\preceq\ z;\ rpr\ A\ r\ \rrbracket \Longrightarrow x\ {}_r\!\prec\ z$
⟨*proof*⟩

lemma *rpr-complete*: $\llbracket\ rpr\ A\ r;\ x \in A;\ y \in A\ \rrbracket \Longrightarrow x\ {}_r\!\preceq\ y \vee y\ {}_r\!\preceq\ x$
⟨*proof*⟩

3.2 Profiles

A *profile* (also termed a collection of *ballots*) maps each individual to an RPR for that individual.

type-synonym $('a,\ 'i)\ Profile = 'i \Rightarrow 'a\ RPR$

definition *profile* $::\ 'a\ set \Rightarrow 'i\ set \Rightarrow ('a,\ 'i)\ Profile \Rightarrow bool$ **where**
$profile\ A\ Is\ P \equiv Is \neq \{\} \wedge (\forall\, i \in Is.\ rpr\ A\ (P\ i))$

lemma *profileI*[*intro*]: $\llbracket\ \bigwedge i.\ i \in Is \Longrightarrow rpr\ A\ (P\ i);\ Is \neq \{\}\ \rrbracket \Longrightarrow profile\ A\ Is\ P$
⟨*proof*⟩

lemma *profile-rprD*[*dest*]: $\llbracket\ profile\ A\ Is\ P;\ i \in Is\ \rrbracket \Longrightarrow rpr\ A\ (P\ i)$
⟨*proof*⟩

lemma *profile-non-empty*: $profile\ A\ Is\ P \Longrightarrow Is \neq \{\}$
⟨*proof*⟩

3.3 Choice Sets, Choice Functions

A *choice set* is the subset of A where every element of that subset is (weakly) preferred to every other element of A with respect to a given RPR. A *choice function* yields a non-empty choice set whenever A is non-empty.

definition $choiceSet :: 'a\ set \Rightarrow 'a\ RPR \Rightarrow 'a\ set$ **where**
$choiceSet\ A\ r \equiv \{\ x \in A\ .\ \forall\, y \in A.\ x\ {}_r\!\preceq\ y\ \}$

definition $choiceFn :: 'a\ set \Rightarrow 'a\ RPR \Rightarrow bool$ **where**
$choiceFn\ A\ r \equiv \forall\, A' \subseteq A.\ A' \neq \{\} \longrightarrow choiceSet\ A'\ r \neq \{\}$

lemma *choiceSetI*[*intro*]:
$[\![\ x \in A;\ \bigwedge y.\ y \in A \Longrightarrow x\ {}_r\!\preceq\ y\ ]\!] \Longrightarrow x \in choiceSet\ A\ r$
⟨*proof*⟩

lemma *choiceFnI*[*intro*]:
$(\bigwedge A'.\ [\![\ A' \subseteq A;\ A' \neq \{\}\ ]\!] \Longrightarrow choiceSet\ A'\ r \neq \{\}) \Longrightarrow choiceFn\ A\ r$
⟨*proof*⟩

If a complete and reflexive relation is also *quasi-transitive* it will yield a choice function.

definition $quasi\text{-}trans :: 'a\ RPR \Rightarrow bool$ **where**
$quasi\text{-}trans\ r \equiv \forall\, x\ y\ z.\ x\ {}_r\!\prec\ y \wedge y\ {}_r\!\prec\ z \longrightarrow x\ {}_r\!\prec\ z$

lemma *quasi-transI*[*intro*]:
$(\bigwedge x\ y\ z.\ [\![\ x\ {}_r\!\prec\ y;\ y\ {}_r\!\prec\ z\ ]\!] \Longrightarrow x\ {}_r\!\prec\ z) \Longrightarrow quasi\text{-}trans\ r$
⟨*proof*⟩

lemma *quasi-transD*: $[\![\ x\ {}_r\!\prec\ y;\ y\ {}_r\!\prec\ z;\ quasi\text{-}trans\ r\ ]\!] \Longrightarrow x\ {}_r\!\prec\ z$
⟨*proof*⟩

lemma *trans-imp-quasi-trans*: $trans\ r \Longrightarrow quasi\text{-}trans\ r$
⟨*proof*⟩

lemma *r-c-qt-imp-cf*:
assumes *finiteA*: *finite A*
and *c*: *complete A r*
and *qt*: *quasi-trans r*
and *r*: *refl-on A r*
shows *choiceFn A r*
⟨*proof*⟩

lemma *rpr-choiceFn*: $[\![\ finite\ A;\ rpr\ A\ r\ ]\!] \Longrightarrow choiceFn\ A\ r$
⟨*proof*⟩

3.4 Social Choice Functions (SCFs)

A *social choice function* (SCF), also called a *collective choice rule* by Sen [Sen70, p28], is a function that somehow aggregates society's opinions, expressed as a profile, into a preference relation.

type-synonym $('a, 'i)$ *SCF* = $('a, 'i)$ *Profile* ⇒ *'a RPR*

The least we require of an SCF is that it be *complete* and some function of the profile. The latter condition is usually implied by other conditions, such as *iia*.

definition
SCF :: $('a, 'i)$ *SCF* ⇒ *'a set* ⇒ *'i set* ⇒ (*'a set* ⇒ *'i set* ⇒ $('a, 'i)$ *Profile* ⇒ *bool*) ⇒ *bool*
where
SCF scf A Is Pcond ≡ (∀ *P*. *Pcond A Is P* ⟶ (*complete A* (*scf P*)))

lemma *SCFI*[*intro*]:
assumes *c*: ⋀*P*. *Pcond A Is P* ⟹ *complete A* (*scf P*)
shows *SCF scf A Is Pcond*
⟨*proof*⟩

lemma *SCF-completeD*[*dest*]: ⟦ *SCF scf A Is Pcond*; *Pcond A Is P* ⟧ ⟹ *complete A* (*scf P*)
⟨*proof*⟩

3.5 Social Welfare Functions (SWFs)

A *Social Welfare Function* (SWF) is an SCF that expresses the society's opinion as a single RPR.

In some situations it might make sense to restrict the allowable profiles.

definition
SWF :: $('a, 'i)$ *SCF* ⇒ *'a set* ⇒ *'i set* ⇒ (*'a set* ⇒ *'i set* ⇒ $('a, 'i)$ *Profile* ⇒ *bool*) ⇒ *bool*
where
SWF swf A Is Pcond ≡ (∀ *P*. *Pcond A Is P* ⟶ *rpr A* (*swf P*))

lemma *SWF-rpr*[*dest*]: ⟦ *SWF swf A Is Pcond*; *Pcond A Is P* ⟧ ⟹ *rpr A* (*swf P*)
⟨*proof*⟩

3.6 General Properties of an SCF

An SCF has a *universal domain* if it works for all profiles.

definition *universal-domain* :: *'a set* ⇒ *'i set* ⇒ $('a, 'i)$ *Profile* ⇒ *bool* **where**
universal-domain A Is P ≡ *profile A Is P*

declare *universal-domain-def*[*simp*]

An SCF is *weakly Pareto-optimal* if, whenever everyone strictly prefers x to y, the SCF does too.

definition
weak-pareto :: $('a, 'i)$ *SCF* ⇒ *'a set* ⇒ *'i set* ⇒ (*'a set* ⇒ *'i set* ⇒ $('a, 'i)$ *Profile* ⇒ *bool*) ⇒ *bool*
where
weak-pareto scf A Is Pcond ≡
(∀ *P x y*. *Pcond A Is P* ∧ $x \in A$ ∧ $y \in A$ ∧ (∀ $i \in$ *Is*. x $_{(P\ i)}\!\prec$ y) ⟶ x $_{(scf\ P)}\!\prec$ y)

lemma *weak-paretoI*[*intro*]:
(⋀*P x y*. ⟦*Pcond A Is P*; $x \in A$; $y \in A$; ⋀*i*. *i*∈*Is* ⟹ x $_{(P\ i)}\!\prec$ y⟧ ⟹ x $_{(scf\ P)}\!\prec$ y)
⟹ *weak-pareto scf A Is Pcond*
⟨*proof*⟩

lemma *weak-paretoD*:
$\llbracket$ *weak-pareto scf A Is Pcond*; *Pcond A Is P*; $x \in A$; $y \in A$;
$(\bigwedge i.\ i \in Is \Longrightarrow x\ _{(P\ i)}{\prec}\ y)$ $\rrbracket \Longrightarrow x\ _{(scf\ P)}{\prec}\ y$
⟨*proof*⟩

An SCF satisfies *independence of irrelevant alternatives* if, for two preference profiles P and P' where for all individuals i, alternatives x and y drawn from set S have the same order in $P\ i$ and $P'\ i$, then alternatives x and y have the same order in *scf* P and *scf* P'.

definition *iia* :: $('a, 'i)$ *SCF* $\Rightarrow$ *'a set* $\Rightarrow$ *'i set* $\Rightarrow$ *bool* **where**
iia scf S Is $\equiv$
$(\forall P\ P'\ x\ y.$ *profile S Is P* $\wedge$ *profile S Is P'*
$\wedge\ x \in S \wedge y \in S$
$\wedge\ (\forall i \in Is.\ ((x\ _{(P\ i)}{\preceq}\ y) \longleftrightarrow (x\ _{(P'\ i)}{\preceq}\ y)) \wedge ((y\ _{(P\ i)}{\preceq}\ x) \longleftrightarrow (y\ _{(P'\ i)}{\preceq}\ x)))$
$\longrightarrow ((x\ _{(scf\ P)}{\preceq}\ y) \longleftrightarrow (x\ _{(scf\ P')}{\preceq}\ y)))$

lemma *iiaI*[*intro*]:
$(\bigwedge P\ P'\ x\ y.$
$\llbracket$ *profile S Is P*; *profile S Is P'*;
$x \in S$; $y \in S$;
$\bigwedge i.\ i \in Is \Longrightarrow ((x\ _{(P\ i)}{\preceq}\ y) \longleftrightarrow (x\ _{(P'\ i)}{\preceq}\ y)) \wedge ((y\ _{(P\ i)}{\preceq}\ x) \longleftrightarrow (y\ _{(P'\ i)}{\preceq}\ x))$
$\rrbracket \Longrightarrow ((x\ _{(swf\ P)}{\preceq}\ y) \longleftrightarrow (x\ _{(swf\ P')}{\preceq}\ y)))$
$\Longrightarrow$ *iia swf S Is*
⟨*proof*⟩

lemma *iiaE*:
$\llbracket$ *iia swf S Is*;
$\{x,y\} \subseteq S$;
$a \in \{x, y\}$; $b \in \{x, y\}$;
$\bigwedge i\ a\ b.\ \llbracket\ a \in \{x, y\};\ b \in \{x, y\};\ i \in Is\ \rrbracket \Longrightarrow (a\ _{(P'\ i)}{\preceq}\ b) \longleftrightarrow (a\ _{(P\ i)}{\preceq}\ b)$;
profile S Is P; *profile S Is P'* $\rrbracket$
$\Longrightarrow (a\ _{(swf\ P)}{\preceq}\ b) \longleftrightarrow (a\ _{(swf\ P')}{\preceq}\ b)$
⟨*proof*⟩

3.7 Decisiveness and Semi-decisiveness

This notion is the key to Arrow's Theorem, and hinges on the use of strict preference [Sen70, p42].

A coalition C of agents is *semi-decisive* for x over y if, whenever the coalition prefers x to y and all other agents prefer the converse, the coalition prevails.

definition *semidecisive* :: $('a, 'i)$ *SCF* $\Rightarrow$ *'a set* $\Rightarrow$ *'i set* $\Rightarrow$ *'i set* $\Rightarrow$ *'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool* **where**
semidecisive scf A Is C x y $\equiv$
$C \subseteq Is \wedge (\forall P.$ *profile A Is P* $\wedge\ (\forall i \in C.\ x\ _{(P\ i)}{\prec}\ y) \wedge (\forall i \in Is - C.\ y\ _{(P\ i)}{\prec}\ x)$
$\longrightarrow x\ _{(scf\ P)}{\prec}\ y)$

lemma *semidecisiveI*[*intro*]:
$\llbracket\ C \subseteq Is$;
$\bigwedge P.\ \llbracket$ *profile A Is P*; $\bigwedge i.\ i \in C \Longrightarrow x\ _{(P\ i)}{\prec}\ y$; $\bigwedge i.\ i \in Is - C \Longrightarrow y\ _{(P\ i)}{\prec}\ x\ \rrbracket$
$\Longrightarrow x\ _{(scf\ P)}{\prec}\ y\ \rrbracket \Longrightarrow$ *semidecisive scf A Is C x y*
⟨*proof*⟩

lemma *semidecisive-coalitionD*[*dest*]: *semidecisive scf A Is C x y* $\Longrightarrow$ *C* $\subseteq$ *Is*
⟨*proof*⟩

lemma *sd-refl*: $[\![$ *C* $\subseteq$ *Is*; *C* $\neq$ {} $]\!]$ $\Longrightarrow$ *semidecisive scf A Is C x x*
⟨*proof*⟩

A coalition C is *decisive* for x over y if, whenever the coalition prefers x to y, the coalition prevails.

definition *decisive* :: ($'a$, $'i$) *SCF* $\Rightarrow$ $'a$ *set* $\Rightarrow$ $'i$ *set* $\Rightarrow$ $'i$ *set* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ $\Rightarrow$ *bool* **where**
decisive scf A Is C x y $\equiv$
$C \subseteq Is \wedge (\forall P.\ profile\ A\ Is\ P \wedge (\forall i \in C.\ x\ _{(P\ i)}{\prec}\ y) \longrightarrow x\ _{(scf\ P)}{\prec}\ y)$

lemma *decisiveI*[*intro*]:
$[\![\ C \subseteq Is;\ \bigwedge P.\ [\![\ profile\ A\ Is\ P;\ \bigwedge i.\ i \in C \Longrightarrow x\ _{(P\ i)}{\prec}\ y\ ]\!] \Longrightarrow x\ _{(scf\ P)}{\prec}\ y\ ]\!]$
$\Longrightarrow$ *decisive scf A Is C x y*
⟨*proof*⟩

lemma *d-imp-sd*: *decisive scf A Is C x y* $\Longrightarrow$ *semidecisive scf A Is C x y*
⟨*proof*⟩

lemma *decisive-coalitionD*[*dest*]: *decisive scf A Is C x y* $\Longrightarrow$ *C* $\subseteq$ *Is*
⟨*proof*⟩

Anyone is trivially decisive for x against x.

lemma *d-refl*: $[\![$ *C* $\subseteq$ *Is*; *C* $\neq$ {} $]\!]$ $\Longrightarrow$ *decisive scf A Is C x x*
⟨*proof*⟩

Agent j is a *dictator* if her preferences always prevail. This is the same as saying that she is decisive for all x and y.

definition *dictator* :: ($'a$, $'i$) *SCF* $\Rightarrow$ $'a$ *set* $\Rightarrow$ $'i$ *set* $\Rightarrow$ $'i$ $\Rightarrow$ *bool* **where**
dictator scf A Is j $\equiv$ $j \in Is \wedge (\forall x \in A.\ \forall y \in A.\ decisive\ scf\ A\ Is\ \{j\}\ x\ y)$

lemma *dictatorI*[*intro*]:
$[\![\ j \in Is;\ \bigwedge x\ y.\ [\![\ x \in A;\ y \in A\ ]\!] \Longrightarrow decisive\ scf\ A\ Is\ \{j\}\ x\ y\ ]\!] \Longrightarrow dictator\ scf\ A\ Is\ j$
⟨*proof*⟩

lemma *dictator-individual*[*dest*]: *dictator scf A Is j* $\Longrightarrow$ $j \in Is$
⟨*proof*⟩

4 Arrow's General Possibility Theorem

The proof falls into two parts: showing that a semi-decisive individual is in fact a dictator, and that a semi-decisive individual exists. I take them in that order.

It might be good to do some of this in a locale. The complication is untangling where various witnesses need to be quantified over.

4.1 Semi-decisiveness Implies Decisiveness

I follow [Sen70, Chapter 3*] quite closely here. Formalising his appeal to the *iia* assumption is the main complication here.

The witness for the first lemma: in the profile P', special agent j strictly prefers x to y to z, and doesn't care about the other alternatives. Everyone else strictly prefers y to each of x to z, and inherits the relative preferences between x and z from profile P.

The model has to be specific about ordering all the other alternatives, but these are immaterial in the proof that uses this witness. Note also that the following lemma is used with different instantiations of x, y and z, so we need to quantify over them here. This happens implicitly, but in a locale we would have to be more explicit.

This is just tedious.

lemma *decisive1-witness*:
  assumes *has3A*: *hasw* $[x,y,z]$ A
    and *profileP*: *profile A Is P*
    and *jIs*: $j \in Is$
  obtains P'
  where *profile A Is* P'
   and $x\ {}_{(P'\ j)}{\prec}\ y \wedge y\ {}_{(P'\ j)}{\prec}\ z$
   and $\bigwedge i.\ i \neq j \Longrightarrow y\ {}_{(P'\ i)}{\prec}\ x \wedge y\ {}_{(P'\ i)}{\prec}\ z \wedge ((x\ {}_{(P'\ i)}{\preceq}\ z) = (x\ {}_{(P\ i)}{\preceq}\ z)) \wedge ((z\ {}_{(P'\ i)}{\preceq}\ x)$
$= (z\ {}_{(P\ i)}{\preceq}\ x))$
⟨*proof*⟩

The key lemma: in the presence of Arrow's assumptions, an individual who is semi-decisive for x and y is actually decisive for x over any other alternative z. (This is where the quantification becomes important.)

lemma *decisive1*:
  assumes *has3A*: *hasw* $[x,y,z]$ A
    and *iia*: *iia swf A Is*
    and *swf*: *SWF swf A Is universal-domain*
    and *wp*: *weak-pareto swf A Is universal-domain*
    and *sd*: *semidecisive swf A Is* $\{j\}$ x y
  shows *decisive swf A Is* $\{j\}$ x z
⟨*proof*⟩

The witness for the second lemma: special agent j strictly prefers z to x to y, and everyone else strictly prefers z to x and y to x. (In some sense the last part is upside-down with respect to the first witness.)

lemma *decisive2-witness*:
  assumes *has3A*: *hasw* $[x,y,z]$ A
    and *profileP*: *profile A Is P*
    and *jIs*: $j \in Is$
  obtains P'
   where *profile A Is* P'
    and $z\ {}_{(P'\ j)}{\prec}\ x \wedge x\ {}_{(P'\ j)}{\prec}\ y$
    and $\bigwedge i.\ i \neq j \Longrightarrow z\ {}_{(P'\ i)}{\prec}\ x \wedge y\ {}_{(P'\ i)}{\prec}\ x \wedge ((y\ {}_{(P'\ i)}{\preceq}\ z) = (y\ {}_{(P\ i)}{\preceq}\ z)) \wedge ((z\ {}_{(P'\ i)}{\preceq}\ y)$
$= (z\ {}_{(P\ i)}{\preceq}\ y))$
⟨*proof*⟩

lemma *decisive2*:
assumes *has3A*: *hasw* [*x*,*y*,*z*] *A*
and *iia*: *iia swf A Is*
and *swf*: *SWF swf A Is universal-domain*
and *wp*: *weak-pareto swf A Is universal-domain*
and *sd*: *semidecisive swf A Is* {*j*} *x y*
shows *decisive swf A Is* {*j*} *z y*
⟨*proof*⟩

The following results permute x, y and z to show how decisiveness can be obtained from semi-decisiveness in all cases. Again, quite tedious.

lemma *decisive3*:
assumes *has3A*: *hasw* [*x*,*y*,*z*] *A*
and *iia*: *iia swf A Is*
and *swf*: *SWF swf A Is universal-domain*
and *wp*: *weak-pareto swf A Is universal-domain*
and *sd*: *semidecisive swf A Is* {*j*} *x z*
shows *decisive swf A Is* {*j*} *y z*
⟨*proof*⟩

lemma *decisive4*:
assumes *has3A*: *hasw* [*x*,*y*,*z*] *A*
and *iia*: *iia swf A Is*
and *swf*: *SWF swf A Is universal-domain*
and *wp*: *weak-pareto swf A Is universal-domain*
and *sd*: *semidecisive swf A Is* {*j*} *y z*
shows *decisive swf A Is* {*j*} *y x*
⟨*proof*⟩

lemma *decisive5*:
assumes *has3A*: *hasw* [*x*,*y*,*z*] *A*
and *iia*: *iia swf A Is*
and *swf*: *SWF swf A Is universal-domain*
and *wp*: *weak-pareto swf A Is universal-domain*
and *sd*: *semidecisive swf A Is* {*j*} *x y*
shows *decisive swf A Is* {*j*} *y x*
⟨*proof*⟩

lemma *decisive6*:
assumes *has3A*: *hasw* [*x*,*y*,*z*] *A*
and *iia*: *iia swf A Is*
and *swf*: *SWF swf A Is universal-domain*
and *wp*: *weak-pareto swf A Is universal-domain*
and *sd*: *semidecisive swf A Is* {*j*} *y x*
shows *decisive swf A Is* {*j*} *y z decisive swf A Is* {*j*} *z x decisive swf A Is* {*j*} *x y*
⟨*proof*⟩

lemma *decisive7*:
assumes *has3A*: *hasw* [*x*,*y*,*z*] *A*
and *iia*: *iia swf A Is*
and *swf*: *SWF swf A Is universal-domain*
and *wp*: *weak-pareto swf A Is universal-domain*
and *sd*: *semidecisive swf A Is* {*j*} *x y*

shows *decisive swf A Is* $\{j\}$ *y z decisive swf A Is* $\{j\}$ *z x decisive swf A Is* $\{j\}$ *x y*
⟨*proof*⟩

lemma *j-decisive-xy*:
assumes *has3A*: *hasw* [*x*,*y*,*z*] *A*
and *iia*: *iia swf A Is*
and *swf*: *SWF swf A Is universal-domain*
and *wp*: *weak-pareto swf A Is universal-domain*
and *sd*: *semidecisive swf A Is* $\{j\}$ *x y*
and *uv*: *hasw* [*u*,*v*] $\{x,y,z\}$
shows *decisive swf A Is* $\{j\}$ *u v*
⟨*proof*⟩

lemma *j-decisive*:
assumes *has3A*: *has 3 A*
and *iia*: *iia swf A Is*
and *swf*: *SWF swf A Is universal-domain*
and *wp*: *weak-pareto swf A Is universal-domain*
and *xyA*: *hasw* [*x*,*y*] *A*
and *sd*: *semidecisive swf A Is* $\{j\}$ *x y*
and *uv*: *hasw* [*u*,*v*] *A*
shows *decisive swf A Is* $\{j\}$ *u v*
⟨*proof*⟩

The first result: if j is semidecisive for some alternatives u and v, then they are actually a dictator.

lemma *sd-imp-dictator*:
assumes *has3A*: *has 3 A*
and *iia*: *iia swf A Is*
and *swf*: *SWF swf A Is universal-domain*
and *wp*: *weak-pareto swf A Is universal-domain*
and *uv*: *hasw* [*u*,*v*] *A*
and *sd*: *semidecisive swf A Is* $\{j\}$ *u v*
shows *dictator swf A Is j*
⟨*proof*⟩

4.2 The Existence of a Semi-decisive Individual

The second half of the proof establishes the existence of a semi-decisive individual. The required witness is essentially an encoding of the Condorcet pardox (aka "the paradox of voting" that shows we get tied up in knots if a certain agent didn't have dictatorial powers.

lemma *sd-exists-witness*:
assumes *has3A*: *hasw* [*x*,*y*,*z*] *A*
and *Vs*: $Is = V1 \cup V2 \cup V3$
$\wedge\ V1 \cap V2 = \{\} \wedge V1 \cap V3 = \{\} \wedge V2 \cap V3 = \{\}$
and *Is*: $Is \neq \{\}$
obtains *P*
where *profile A Is P*
and $\forall\, i \in V1.\ x\ {}_{(P\ i)}{\prec}\ y \wedge y\ {}_{(P\ i)}{\prec}\ z$
and $\forall\, i \in V2.\ z\ {}_{(P\ i)}{\prec}\ x \wedge x\ {}_{(P\ i)}{\prec}\ y$
and $\forall\, i \in V3.\ y\ {}_{(P\ i)}{\prec}\ z \wedge z\ {}_{(P\ i)}{\prec}\ x$
⟨*proof*⟩

This proof is unfortunately long. Many of the statements rely on a lot of context, making it difficult to split it up.

lemma *sd-exists*:
  assumes *has3A*: *has 3 A*
    and *finiteIs*: *finite Is*
    and *twoIs*: *has 2 Is*
    and *iia*: *iia swf A Is*
    and *swf*: *SWF swf A Is universal-domain*
    and *wp*: *weak-pareto swf A Is universal-domain*
  shows $\exists j\ u\ v.\ hasw\ [u,v]\ A \wedge semidecisive\ swf\ A\ Is\ \{j\}\ u\ v$
⟨*proof*⟩

4.3 Arrow's General Possibility Theorem

Finally we conclude with the celebrated "possibility" result. Note that we assume the set of individuals is finite; [Rou79] relaxes this with some fancier set theory. Having an infinite set of alternatives doesn't matter, though the result is a bit more plausible if we assume finiteness [Sen70, p54].

theorem *ArrowGeneralPossibility*:
  assumes *has3A*: *has 3 A*
    and *finiteIs*: *finite Is*
    and *has2Is*: *has 2 Is*
    and *iia*: *iia swf A Is*
    and *swf*: *SWF swf A Is universal-domain*
    and *wp*: *weak-pareto swf A Is universal-domain*
  obtains *j* **where** *dictator swf A Is j*
  ⟨*proof*⟩

5 Sen's Liberal Paradox

5.1 Social Decision Functions (SDFs)

To make progress in the face of Arrow's Theorem, the demands placed on the social choice function need to be weakened. One approach is to only require that the set of alternatives that society ranks highest (and is otherwise indifferent about) be non-empty.

Following [Sen70, Chapter 4*], a *Social Decision Function* (SDF) yields a choice function for every profile.

definition
  $SDF :: ('a, 'i)\ SCF \Rightarrow 'a\ set \Rightarrow 'i\ set \Rightarrow ('a\ set \Rightarrow 'i\ set \Rightarrow ('a, 'i)\ Profile \Rightarrow bool) \Rightarrow bool$
where
  $SDF\ sdf\ A\ Is\ Pcond \equiv (\forall P.\ Pcond\ A\ Is\ P \longrightarrow choiceFn\ A\ (sdf\ P))$

lemma *SDFI*[*intro*]:
  $(\bigwedge P.\ Pcond\ A\ Is\ P \Longrightarrow choiceFn\ A\ (sdf\ P)) \Longrightarrow SDF\ sdf\ A\ Is\ Pcond$
  ⟨*proof*⟩

lemma *SWF-SDF*:
  assumes *finiteA*: *finite A*

shows *SWF scf A Is universal-domain* $\Longrightarrow$ *SDF scf A Is universal-domain*
⟨*proof*⟩

In contrast to SWFs, there are SDFs satisfying Arrow's (relevant) requirements. The lemma uses a witness to show the absence of a dictatorship.

lemma *SDF-nodictator-witness*:
assumes *has2A*: *hasw* [*x*,*y*] *A*
and *has2Is*: *hasw* [*i*,*j*] *Is*
obtains *P*
where *profile A Is P*
and $x\ _{(P\ i)}\!\prec y$
and $y\ _{(P\ j)}\!\prec x$
⟨*proof*⟩

lemma *SDF-possibility*:
assumes *finiteA*: *finite A*
and *has2A*: *has 2 A*
and *has2Is*: *has 2 Is*
obtains *sdf*
where *weak-pareto sdf A Is universal-domain*
and *iia sdf A Is*
and $\neg(\exists j.$ *dictator sdf A Is j*$)$
and *SDF sdf A Is universal-domain*
⟨*proof*⟩

Sen makes several other stronger statements about SDFs later in the chapter. I leave these for future work.

5.2 Sen's Liberal Paradox

Having side-stepped Arrow's Theorem, Sen proceeds to other conditions one may ask of an SCF. His analysis of *liberalism*, mechanised in this section, has attracted much criticism over the years [AK96].

Following [Sen70, Chapter 6*], a *liberal* social choice rule is one that, for each individual, there is a pair of alternatives that she is decisive over.

definition *liberal* :: ('*a*, '*i*) *SCF* ⇒ '*a set* ⇒ '*i set* ⇒ *bool* **where**
liberal scf A Is ≡
$(\forall i \in Is.\ \exists x \in A.\ \exists y \in A.\ x \neq y$
$\wedge$ *decisive scf A Is* $\{i\}$ *x y* $\wedge$ *decisive scf A Is* $\{i\}$ *y x*$)$

lemma *liberalE*:
⟦ *liberal scf A Is*; $i \in Is$ ⟧
$\Longrightarrow \exists x \in A.\ \exists y \in A.\ x \neq y$
$\wedge$ *decisive scf A Is* $\{i\}$ *x y* $\wedge$ *decisive scf A Is* $\{i\}$ *y x*
⟨*proof*⟩

This condition can be weakened to require just two such decisive individuals; if we required just one, we would allow dictatorships, which are clearly not liberal.

definition *minimally-liberal* :: ('*a*, '*i*) *SCF* ⇒ '*a set* ⇒ '*i set* ⇒ *bool* **where**
minimally-liberal scf A Is ≡
$(\exists i \in Is.\ \exists j \in Is.\ i \neq j$

$\wedge$ ($\exists\, x \in A.\ \exists\, y \in A.\ x \neq y$
$\wedge$ *decisive scf A Is* $\{i\}$ *x y* $\wedge$ *decisive scf A Is* $\{i\}$ *y x*)
$\wedge$ ($\exists\, x \in A.\ \exists\, y \in A.\ x \neq y$
$\wedge$ *decisive scf A Is* $\{j\}$ *x y* $\wedge$ *decisive scf A Is* $\{j\}$ *y x*))

lemma *liberal-imp-minimally-liberal*:
assumes *has2Is*: *has 2 Is*
and *L*: *liberal scf A Is*
shows *minimally-liberal scf A Is*
⟨*proof*⟩

The key observation is that once we have at least two decisive individuals we can complete the Condorcet (paradox of voting) cycle using the weak Pareto assumption. The details of the proof don't give more insight.

Firstly we need three types of profile witnesses (one of which we saw previously). The main proof proceeds by case distinctions on which alternatives the two liberal agents are decisive for.

lemmas *liberal-witness-two* = *SDF-nodictator-witness*

lemma *liberal-witness-three*:
assumes *threeA*: *hasw* [*x*,*y*,*v*] *A*
and *twoIs*: *hasw* [*i*,*j*] *Is*
obtains *P*
where *profile A Is P*
and $x\ {}_{(P\ i)}{\prec}\ y$
and $v\ {}_{(P\ j)}{\prec}\ x$
and $\forall\, i \in Is.\ y\ {}_{(P\ i)}{\prec}\ v$
⟨*proof*⟩

lemma *liberal-witness-four*:
assumes *fourA*: *hasw* [*x*,*y*,*u*,*v*] *A*
and *twoIs*: *hasw* [*i*,*j*] *Is*
obtains *P*
where *profile A Is P*
and $x\ {}_{(P\ i)}{\prec}\ y$
and $u\ {}_{(P\ j)}{\prec}\ v$
and $\forall\, i \in Is.\ v\ {}_{(P\ i)}{\prec}\ x \wedge y\ {}_{(P\ i)}{\prec}\ u$
⟨*proof*⟩

The Liberal Paradox: having two decisive individuals, an SDF and the weak pareto assumption is inconsistent.

theorem *LiberalParadox*:
assumes *SDF*: *SDF sdf A Is universal-domain*
and *ml*: *minimally-liberal sdf A Is*
and *wp*: *weak-pareto sdf A Is universal-domain*
shows *False*
⟨*proof*⟩

6 May's Theorem

May's Theorem [May52] provides a characterisation of majority voting in terms of four conditions that appear quite natural for *a priori* unbiased social choice scenarios. It can be seen as a refinement of some earlier work by Arrow [Arr63, Chapter V.1].

The following is a mechanisation of Sen's generalisation [Sen70, Chapter 5*]; originally Arrow and May consider only two alternatives, whereas Sen's model maps profiles of full RPRs to a possibly intransitive relation that does at least generate a choice set that satisfies May's conditions.

6.1 May's Conditions

The condition of *anonymity* asserts that the individuals' identities are not considered by the choice rule. Rather than talk about permutations we just assert the result of the SCF is the same when the profile is composed with an arbitrary bijection on the set of individuals.

definition *anonymous* :: $('a,\ 'i)\ SCF \Rightarrow 'a\ set \Rightarrow 'i\ set \Rightarrow bool$ **where**
  anonymous scf A Is $\equiv$
    $(\forall P\ f\ x\ y.\ profile\ A\ Is\ P \wedge bij\text{-}betw\ f\ Is\ Is \wedge x \in A \wedge y \in A$
      $\longrightarrow (x\ _{(scf\ P)}\preceq\ y) = (x\ _{(scf\ (P \circ f))}\preceq\ y))$

lemma *anonymousI*[*intro*]:
  $(\bigwedge P\ f\ x\ y.\ [\![\ profile\ A\ Is\ P;\ bij\text{-}betw\ f\ Is\ Is;$
      $x \in A;\ y \in A\ ]\!] \Longrightarrow (x\ _{(scf\ P)}\preceq\ y) = (x\ _{(scf\ (P \circ f))}\preceq\ y))$
  $\Longrightarrow$ *anonymous scf A Is*
  ⟨*proof*⟩

lemma *anonymousD*:
  $[\![\ anonymous\ scf\ A\ Is;\ profile\ A\ Is\ P;\ bij\text{-}betw\ f\ Is\ Is;\ x \in A;\ y \in A\ ]\!]$
  $\Longrightarrow (x\ _{(scf\ P)}\preceq\ y) = (x\ _{(scf\ (P \circ f))}\preceq\ y)$
  ⟨*proof*⟩

Similarly, an SCF is *neutral* if it is insensitive to the identity of the alternatives. This is Sen's characterisation [Sen70, p72].

definition *neutral* :: $('a,\ 'i)\ SCF \Rightarrow 'a\ set \Rightarrow 'i\ set \Rightarrow bool$ **where**
  neutral scf A Is $\equiv$
    $(\forall P\ P'\ x\ y\ z\ w.\ profile\ A\ Is\ P \wedge profile\ A\ Is\ P' \wedge x \in A \wedge y \in A \wedge z \in A \wedge w \in A$
    $\wedge (\forall i \in Is.\ x\ _{(P\ i)}\preceq\ y \longleftrightarrow z\ _{(P'\ i)}\preceq\ w) \wedge (\forall i \in Is.\ y\ _{(P\ i)}\preceq\ x \longleftrightarrow w\ _{(P'\ i)}\preceq\ z)$
  $\longrightarrow ((x\ _{(scf\ P)}\preceq\ y \longleftrightarrow z\ _{(scf\ P')}\preceq\ w) \wedge (y\ _{(scf\ P)}\preceq\ x \longleftrightarrow w\ _{(scf\ P')}\preceq\ z)))$

lemma *neutralI*[*intro*]:
  $(\bigwedge P\ P'\ x\ y\ z\ w.$
    $[\![\ profile\ A\ Is\ P;\ profile\ A\ Is\ P';\ \{x,y,z,w\} \subseteq A;$
      $\bigwedge i.\ i \in Is \Longrightarrow x\ _{(P\ i)}\preceq\ y \longleftrightarrow z\ _{(P'\ i)}\preceq\ w;$
      $\bigwedge i.\ i \in Is \Longrightarrow y\ _{(P\ i)}\preceq\ x \longleftrightarrow w\ _{(P'\ i)}\preceq\ z\ ]\!]$
    $\Longrightarrow ((x\ _{(scf\ P)}\preceq\ y \longleftrightarrow z\ _{(scf\ P')}\preceq\ w) \wedge (y\ _{(scf\ P)}\preceq\ x \longleftrightarrow w\ _{(scf\ P')}\preceq\ z)))$
  $\Longrightarrow$ *neutral scf A Is*
  ⟨*proof*⟩

lemma *neutralD*:
  $[\![\ neutral\ scf\ A\ Is;$

$\quad$ *profile A Is P*; *profile A Is P'*; $\{x,y,z,w\} \subseteq A$;
$\quad \bigwedge i.\ i \in Is \Longrightarrow x\ {}_{(P\ i)}\!\preceq y \longleftrightarrow z\ {}_{(P'\ i)}\!\preceq w$;
$\quad \bigwedge i.\ i \in Is \Longrightarrow y\ {}_{(P\ i)}\!\preceq x \longleftrightarrow w\ {}_{(P'\ i)}\!\preceq z\ \rrbracket$
$\Longrightarrow (x\ {}_{(scf\ P)}\!\preceq y \longleftrightarrow z\ {}_{(scf\ P')}\!\preceq w) \wedge (y\ {}_{(scf\ P)}\!\preceq x \longleftrightarrow w\ {}_{(scf\ P')}\!\preceq z)$
⟨*proof*⟩

Neutrality implies independence of irrelevant alternatives.

lemma *neutral-iia*: *neutral scf A Is* $\Longrightarrow$ *iia scf A Is*
⟨*proof*⟩

Positive responsiveness is a bit like non-manipulability: if one individual improves their opinion of x, then the result should shift in favour of x.

definition *positively-responsive* :: $('a,\ 'i)\ SCF \Rightarrow\ 'a\ set \Rightarrow\ 'i\ set \Rightarrow bool$ **where**
positively-responsive scf A Is $\equiv$
$(\forall P\ P'\ x\ y.\ profile\ A\ Is\ P \wedge profile\ A\ Is\ P' \wedge x \in A \wedge y \in A$
$\wedge (\forall i \in Is.\ (x\ {}_{(P\ i)}\!\prec y \longrightarrow x\ {}_{(P'\ i)}\!\prec y) \wedge (x\ {}_{(P\ i)}\!\approx y \longrightarrow x\ {}_{(P'\ i)}\!\preceq y))$
$\wedge (\exists k \in Is.\ (x\ {}_{(P\ k)}\!\approx y \wedge x\ {}_{(P'\ k)}\!\prec y) \vee (y\ {}_{(P\ k)}\!\prec x \wedge x\ {}_{(P'\ k)}\!\preceq y))$
$\longrightarrow x\ {}_{(scf\ P)}\!\preceq y \longrightarrow x\ {}_{(scf\ P')}\!\prec y)$

lemma *positively-responsiveI*[*intro*]:
assumes I: $\bigwedge P\ P'\ x\ y.$
$\llbracket$ *profile A Is P*; *profile A Is P'*; $x \in A$; $y \in A$;
$\bigwedge i.\ \llbracket\ i \in Is;\ x\ {}_{(P\ i)}\!\prec y\ \rrbracket \Longrightarrow x\ {}_{(P'\ i)}\!\prec y$;
$\bigwedge i.\ \llbracket\ i \in Is;\ x\ {}_{(P\ i)}\!\approx y\ \rrbracket \Longrightarrow x\ {}_{(P'\ i)}\!\preceq y$;
$\exists k \in Is.\ (x\ {}_{(P\ k)}\!\approx y \wedge x\ {}_{(P'\ k)}\!\prec y) \vee (y\ {}_{(P\ k)}\!\prec x \wedge x\ {}_{(P'\ k)}\!\preceq y)$;
$x\ {}_{(scf\ P)}\!\preceq y\ \rrbracket$
$\Longrightarrow x\ {}_{(scf\ P')}\!\prec y$
shows *positively-responsive scf A Is*
⟨*proof*⟩

lemma *positively-responsiveD*:
$\llbracket$ *positively-responsive scf A Is*;
profile A Is P; *profile A Is P'*; $x \in A$; $y \in A$;
$\bigwedge i.\ \llbracket\ i \in Is;\ x\ {}_{(P\ i)}\!\prec y\ \rrbracket \Longrightarrow x\ {}_{(P'\ i)}\!\prec y$;
$\bigwedge i.\ \llbracket\ i \in Is;\ x\ {}_{(P\ i)}\!\approx y\ \rrbracket \Longrightarrow x\ {}_{(P'\ i)}\!\preceq y$;
$\exists k \in Is.\ (x\ {}_{(P\ k)}\!\approx y \wedge x\ {}_{(P'\ k)}\!\prec y) \vee (y\ {}_{(P\ k)}\!\prec x \wedge x\ {}_{(P'\ k)}\!\preceq y)$;
$x\ {}_{(scf\ P)}\!\preceq y\ \rrbracket$
$\Longrightarrow x\ {}_{(scf\ P')}\!\prec y$
⟨*proof*⟩

6.2 The Method of Majority Decision satisfies May's conditions

The *method of majority decision* (MMD) says that if the number of individuals who strictly prefer x to y is larger than or equal to those who strictly prefer the converse, then $x\ R\ y$. Note that this definition only makes sense for a finite population.

definition *MMD* :: $'i\ set \Rightarrow ('a,\ 'i)\ SCF$ **where**
$MMD\ Is\ P \equiv \{\ (x,\ y)\ .\ card\ \{\ i \in Is.\ x\ {}_{(P\ i)}\!\prec y\ \} \geq card\ \{\ i \in Is.\ y\ {}_{(P\ i)}\!\prec x\ \}\ \}$

The first part of May's Theorem establishes that the conditions are consistent, by showing that they are satisfied by MMD.

lemma *MMD-l2r*:
  fixes $A :: 'a\ set$
    and $Is :: 'i\ set$
  assumes *finiteIs*: *finite Is*
  shows *SCF* (*MMD Is*) *A Is universal-domain*
    and *anonymous* (*MMD Is*) *A Is*
    and *neutral* (*MMD Is*) *A Is*
    and *positively-responsive* (*MMD Is*) *A Is*
⟨*proof*⟩

6.3 Everything satisfying May's conditions is the Method of Majority Decision

Now show that MMD is the only SCF that satisfies these conditions.

Firstly develop some theory about exchanging alternatives x and y in profile P.

definition $swapAlts :: 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a$ **where**
  $swapAlts\ a\ b\ u \equiv if\ u = a\ then\ b\ else\ if\ u = b\ then\ a\ else\ u$

lemma *swapAlts-in-set-iff*: $\{a,\ b\} \subseteq A \Longrightarrow swapAlts\ a\ b\ u \in A \longleftrightarrow u \in A$
  ⟨*proof*⟩

definition $swapAltsP :: ('a,\ 'i)\ Profile \Rightarrow 'a \Rightarrow 'a \Rightarrow ('a,\ 'i)\ Profile$ **where**
  $swapAltsP\ P\ a\ b \equiv (\lambda i.\ \{\ (u,\ v)\ .\ (swapAlts\ a\ b\ u,\ swapAlts\ a\ b\ v) \in P\ i\ \})$

lemma *swapAltsP-ab*: $a\ {}_{(P\ i)}{\preceq}\ b \longleftrightarrow b\ {}_{(swapAltsP\ P\ a\ b\ i)}{\preceq}\ a\ b\ {}_{(P\ i)}{\preceq}\ a \longleftrightarrow a\ {}_{(swapAltsP\ P\ a\ b\ i)}{\preceq}$
b
  ⟨*proof*⟩

lemma *profile-swapAltsP*:
  assumes *profileP*: *profile A Is P*
      and *abA*: $\{a,b\} \subseteq A$
  shows *profile A Is* (*swapAltsP P a b*)
⟨*proof*⟩

lemma *profile-bij-profile*:
  assumes *profileP*: *profile A Is P*
      and *bijf*: *bij-betw f Is Is*
  shows *profile A Is* $(P \circ f)$
  ⟨*proof*⟩

The locale keeps the conditions in scope for the next few lemmas. Note how weak the constraints on the sets of alternatives and individuals are; clearly there needs to be at least two alternatives and two individuals for conflict to occur, but it is pleasant that the proof uniformly handles the degenerate cases.

locale *May* =
  fixes $A :: 'a\ set$

  fixes $Is :: 'i\ set$
  assumes *finiteIs*: *finite Is*

  fixes $scf :: ('a,\ 'i)\ SCF$

assumes *SCF*: *SCF scf A Is universal-domain*
and *anonymous*: *anonymous scf A Is*
and *neutral*: *neutral scf A Is*
and *positively-responsive*: *positively-responsive scf A Is*
begin

Anonymity implies that, for any pair of alternatives, the social choice rule can only depend on the number of individuals who express any given preference between them. Note we also need *iia*, implied by neutrality, to restrict attention to alternatives x and y.

lemma *anonymous-card*:
assumes *profileP*: *profile A Is P*
and *profileP'*: *profile A Is P'*
and *xyA*: *hasw* [x,y] *A*
and *xytally*: $card\ \{\ i \in Is.\ x\ _{(P\ i)}\prec\ y\ \} = card\ \{\ i \in Is.\ x\ _{(P'\ i)}\prec\ y\ \}$
and *yxtally*: $card\ \{\ i \in Is.\ y\ _{(P\ i)}\prec\ x\ \} = card\ \{\ i \in Is.\ y\ _{(P'\ i)}\prec\ x\ \}$
shows $x\ _{(scf\ P)}\preceq\ y \longleftrightarrow x\ _{(scf\ P')}\preceq\ y$
⟨*proof*⟩

Using the previous result and neutrality, it must be the case that if the tallies are tied for alternatives x and y then the social choice function is indifferent between those two alternatives.

lemma *anonymous-neutral-indifference*:
assumes *profileP*: *profile A Is P*
and *xyA*: *hasw* [x,y] *A*
and *tallyP*: $card\ \{\ i \in Is.\ x\ _{(P\ i)}\prec\ y\ \} = card\ \{\ i \in Is.\ y\ _{(P\ i)}\prec\ x\ \}$
shows $x\ _{(scf\ P)}\approx\ y$
⟨*proof*⟩

Finally, if the tallies are not equal then the social choice function must lean towards the one with the higher count due to positive responsiveness.

lemma *positively-responsive-prefer-witness*:
assumes *profileP*: *profile A Is P*
and *xyA*: *hasw* [x,y] *A*
and *tallyP*: $card\ \{\ i \in Is.\ x\ _{(P\ i)}\prec\ y\ \} > card\ \{\ i \in Is.\ y\ _{(P\ i)}\prec\ x\ \}$
obtains $P'\ k$
where *profile A Is P'*
and $\bigwedge i.\ [\![i \in Is;\ x\ _{(P'\ i)}\prec\ y]\!] \Longrightarrow x\ _{(P\ i)}\prec\ y$
and $\bigwedge i.\ [\![i \in Is;\ x\ _{(P'\ i)}\approx\ y]\!] \Longrightarrow x\ _{(P\ i)}\preceq\ y$
and $k \in Is \wedge x\ _{(P'\ k)}\approx\ y \wedge x\ _{(P\ k)}\prec\ y$
and $card\ \{\ i \in Is.\ x\ _{(P'\ i)}\prec\ y\ \} = card\ \{\ i \in Is.\ y\ _{(P'\ i)}\prec\ x\ \}$
⟨*proof*⟩

lemma *positively-responsive-prefer*:
assumes *profileP*: *profile A Is P*
and *xyA*: *hasw* [x,y] *A*
and *tallyP*: $card\ \{\ i \in Is.\ x\ _{(P\ i)}\prec\ y\ \} > card\ \{\ i \in Is.\ y\ _{(P\ i)}\prec\ x\ \}$
shows $x\ _{(scf\ P)}\prec\ y$
⟨*proof*⟩

lemma *MMD-r2l*:

assumes *profileP*: *profile A Is P*
  and *xyA*: *hasw* [*x*,*y*] *A*
shows $x\ _{(scf\ P)}\preceq\ y \longleftrightarrow x\ _{(MMD\ Is\ P)}\preceq\ y$
⟨*proof*⟩

end

May's original paper [May52] goes on to show that the conditions are independent by exhibiting choice rules that differ from *MMD* and satisfy the conditions remaining after any particular one is removed. I leave this to future work.

May also wrote a later article [May53] where he shows that the conditions are completely independent, i.e. for every partition of the conditions into two sets, there is a voting rule that satisfies one and not the other.

There are many later papers that characterise MMD with different sets of conditions.

6.4 The Plurality Rule

Goodin and List [GL06] show that May's original result can be generalised to characterise plurality voting. The following shows that this result is a short step from Sen's much earlier generalisation.

Plurality voting is a choice function that returns the alternative that receives the most votes, or the set of such alternatives in the case of a tie. Profiles are restricted to those where each individual casts a vote in favour of a single alternative.

type-synonym $('a,\ 'i)\ SVProfile = \ 'i \Rightarrow \ 'a$

definition *svprofile* :: $'a\ set \Rightarrow \ 'i\ set \Rightarrow ('a,\ 'i)\ SVProfile \Rightarrow bool$ **where**
  $svprofile\ A\ Is\ F \equiv Is \neq \{\} \wedge F\ `\ Is \subseteq A$

definition *plurality-rule* :: $'a\ set \Rightarrow \ 'i\ set \Rightarrow ('a,\ 'i)\ SVProfile \Rightarrow \ 'a\ set$ **where**
  plurality-rule A Is F
    $\equiv \{\ x \in A\ .\ \forall y \in A.\ card\ \{\ i \in Is\ .\ F\ i = x\ \} \geq card\ \{\ i \in Is\ .\ F\ i = y\ \}\ \}$

By translating single-vote profiles into RPRs in the obvious way, the choice function arising from *MMD* coincides with traditional plurality voting.

definition *MMD-plurality-rule* :: $'a\ set \Rightarrow \ 'i\ set \Rightarrow ('a,\ 'i)\ Profile \Rightarrow \ 'a\ set$ **where**
  MMD-plurality-rule A Is P ≡ *choiceSet A* (*MMD Is P*)

definition *single-vote-to-RPR* :: $'a\ set \Rightarrow \ 'a \Rightarrow \ 'a\ RPR$ **where**
  $single\text{-}vote\text{-}to\text{-}RPR\ A\ a \equiv \{\ (a,\ x)\ |x.\ x \in A\ \} \cup (A - \{a\}) \times (A - \{a\})$

lemma *single-vote-to-RPR-iff*:
  $[\![\ a \in A;\ x \in A;\ a \neq x\ ]\!] \Longrightarrow (a\ _{(single\text{-}vote\text{-}to\text{-}RPR\ A\ b)}\prec\ x) \longleftrightarrow (b = a)$
  ⟨*proof*⟩

lemma *plurality-rule-equiv*:
  plurality-rule A Is F = *MMD-plurality-rule A Is* (*single-vote-to-RPR A* ∘ *F*)
⟨*proof*⟩

Thus it is clear that Sen's generalisation of May's result applies to this case as well.

Their paper goes on to show how strengthening the anonymity condition gives rise to a characterisation of approval voting that strictly generalises May's original theorem. As this requires some rearrangement of the proof I leave it to future work.

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