Relational Characterisations of Paths

Walter Guttmann and Peter Höfner

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Abstract

Binary relations are one of the standard ways to encode, characterise and reason about graphs. Relation algebras provide equational axioms for a large fragment of the calculus of binary relations. Although relations are standard tools in many areas of mathematics and computing, researchers usually fall back to point-wise reasoning when it comes to arguments about paths in a graph. We present a purely algebraic way to specify different kinds of paths in Kleene relation algebras, which are relation algebras equipped with an operation for reflexive transitive closure. We study the relationship between paths with a designated root vertex and paths without such a vertex. Since we stay in first-order logic this development helps with mechanising proofs. To demonstrate the applicability of the algebraic framework we verify the correctness of three basic graph algorithms.

Contents

Overview

A path in a graph can be defined as a connected subgraph of edges where each vertex has at most one incoming edge and at most one outgoing edge [3, 12]. We develop a theory of paths based on this representation and use it for algorithm verification. All reasoning is done in variants of relation algebras and Kleene algebras [8, 9, 11].

Section 1 presents fundamental results that hold in relation algebras. Relation-algebraic characterisations of various kinds of paths are introduced and compared in Section 2. We extend this to paths with a designated root in Section 3. Section 4 verifies the correctness of a few basic graph algorithms.

These Isabelle/HOL theories formally verify results in [2]. See this paper for further details and related work.

1 (More) Relation Algebra

This theory presents fundamental properties of relation algebras, which are not present in the AFP entry on relation algebras but could be integrated there [1]. Many theorems concern vectors and points.

theory *More-Relation-Algebra*

imports *Relation-Algebra.Relation-Algebra-RTC*
Relation-Algebra.Relation-Algebra-Functions

begin

unbundle *no trancl-syntax*

context *relation-algebra*
begin

notation
  converse (‹(-^{T})› [*102*] *101*)

abbreviation *bijective*
  where *bijective* $x \equiv$ *is-inj* $x \wedge$ *is-sur* x

abbreviation *reflexive*
  where *reflexive* $R \equiv 1' \leq R$

abbreviation *symmetric*
  where *symmetric* $R \equiv R = R^T$

abbreviation *transitive*
  where *transitive* $R \equiv R;R \leq R$

General theorems

lemma *x-leq-triple-x*:
$x \leq x;x^T;x$
⟨*proof*⟩

lemma *inj-triple*:
assumes *is-inj* x
shows $x = x;x^T;x$
⟨*proof*⟩

lemma *p-fun-triple*:
assumes *is-p-fun* x
shows $x = x;x^T;x$
⟨*proof*⟩

lemma *loop-backward-forward*:
$x^T \leq -(1') + x$
⟨*proof*⟩

lemma *inj-sur-semi-swap*:
assumes *is-sur* z
and *is-inj* x
shows $z \leq y;x \Longrightarrow x \leq y^T;z$
⟨*proof*⟩

lemma *inj-sur-semi-swap-short*:
assumes *is-sur* z
and *is-inj* x
shows $z \leq y^T;x \Longrightarrow x \leq y;z$
⟨*proof*⟩

lemma *bij-swap*:
assumes *bijective* z
and *bijective* x
shows $z \leq y^T;x \longleftrightarrow x \leq y;z$
⟨*proof*⟩

The following result is [10, Proposition 4.2.2(iv)].

lemma *ss422iv*:
assumes *is-p-fun* y
and $x \leq y$
and $y;1 \leq x;1$
shows $x = y$
⟨*proof*⟩

The following results are variants of [10, Proposition 4.2.3].

lemma *ss423conv*:
assumes *bijective* x
shows $x ; y \leq z \longleftrightarrow y \leq x^T ; z$

⟨*proof*⟩

lemma *ss423bij*:
assumes *bijective* x
shows $y ; x^T \leq z \longleftrightarrow y \leq z ; x$
⟨*proof*⟩

lemma *inj-distr*:
assumes *is-inj* z
shows $(x \cdot y);z = (x;z) \cdot (y;z)$
⟨*proof*⟩

lemma *test-converse*:
$x \cdot 1' = x^T \cdot 1'$
⟨*proof*⟩

lemma *injective-down-closed*:
assumes *is-inj* x
and $y \leq x$
shows *is-inj* y
⟨*proof*⟩

lemma *injective-sup*:
assumes *is-inj* t
and $e;t^T \leq 1'$
and *is-inj* e
shows *is-inj* $(t + e)$
⟨*proof*⟩

Some (more) results about vectors

lemma *vector-meet-comp*:
assumes *is-vector* v
and *is-vector* w
shows $v;w^T = v \cdot w^T$
⟨*proof*⟩

lemma *vector-meet-comp′*:
assumes *is-vector* v
shows $v;v^T = v \cdot v^T$
⟨*proof*⟩

lemma *vector-meet-comp-x*:
$x;1;x^T = x;1 \cdot 1;x^T$
⟨*proof*⟩

lemma *vector-meet-comp-x′*:
$x;1;x = x;1 \cdot 1;x$
⟨*proof*⟩

lemma *vector-prop1*:
  assumes *is-vector* v
    shows $-v^T;v = 0$
⟨*proof*⟩

The following results and a number of others in this theory are from [5].

lemma *ee*:
  assumes *is-vector* v
    and $e \leq v;-v^T$
  shows $e;e = 0$
⟨*proof*⟩

lemma *et*:
  assumes *is-vector* v
    and $e \leq v;-v^T$
    and $t \leq v;v^T$
  shows $e;t = 0$
    and $e;t^T = 0$
⟨*proof*⟩

Some (more) results about points

definition *point*
  where *point* $x \equiv$ *is-vector* $x \wedge$ *bijective* x

lemma *point-swap*:
  assumes *point* p
    and *point* q
  shows $p \leq x;q \longleftrightarrow q \leq x^T;p$
⟨*proof*⟩

Some (more) results about singletons

abbreviation *singleton*
  where *singleton* $x \equiv$ *bijective* $(x;1) \wedge$ *bijective* $(x^T;1)$

lemma *singleton-injective*:
  assumes *singleton* x
  shows *is-inj* x
⟨*proof*⟩

lemma *injective-inv*:
  assumes *is-vector* v
    and *singleton* e
    and $e \leq v;-v^T$
    and $t \leq v;v^T$
    and *is-inj* t
  shows *is-inj* $(t + e)$
⟨*proof*⟩

lemma *singleton-is-point*:

assumes *singleton p*
shows *point* $(p;1)$
⟨*proof*⟩

lemma *singleton-transp*:
assumes *singleton p*
shows *singleton* (p^T)
⟨*proof*⟩

lemma *point-to-singleton*:
assumes *singleton p*
shows *singleton* $(1'{\cdot}p;p^T)$
⟨*proof*⟩

lemma *singleton-singletonT*:
assumes *singleton p*
shows $p;p^T \leq 1'$
⟨*proof*⟩

Minimality

abbreviation *minimum*
where *minimum x v* $\equiv v \cdot -(x^T;v)$

Regressively finite

abbreviation *regressively-finite*
where *regressively-finite x* $\equiv \forall\, v\ .\ \textit{is-vector}\ v \wedge v \leq x^T;v \longrightarrow v = 0$

lemma *regressively-finite-minimum*:
regressively-finite R $\Longrightarrow$ *is-vector v* $\Longrightarrow v \neq 0 \Longrightarrow$ *minimum R v* $\neq 0$
⟨*proof*⟩

lemma *regressively-finite-irreflexive*:
assumes *regressively-finite x*
shows $x \leq -1'$
⟨*proof*⟩

end

1.1 Relation algebras satisfying the Tarski rule

class *relation-algebra-tarski* = *relation-algebra* +
assumes *tarski*: $x \neq 0 \longleftrightarrow 1;x;1 = 1$
begin

Some (more) results about points

lemma *point-equations*:
assumes *is-point p*
shows $p;1=p$
and $1;p=1$

and $p^T;1=1$
and $1;p^T=p^T$
$\langle proof \rangle$

The following result is [10, Proposition 2.4.5(i)].

lemma *point-singleton*:
assumes *is-point p*
and *is-vector v*
and $v \neq 0$
and $v \leq p$
shows $v = p$
$\langle proof \rangle$

lemma *point-not-equal-aux*:
assumes *is-point p*
and *is-point q*
shows $p \neq q \longleftrightarrow p \cdot -q \neq 0$
$\langle proof \rangle$

The following result is part of [10, Proposition 2.4.5(ii)].

lemma *point-not-equal*:
assumes *is-point p*
and *is-point q*
shows $p \neq q \longleftrightarrow p \leq -q$
and $p \leq -q \longleftrightarrow p;q^T \leq -1'$
and $p;q^T \leq -1' \longleftrightarrow p^T;q \leq 0$
$\langle proof \rangle$

lemma *point-is-point*:
point x $\longleftrightarrow$ *is-point x*
$\langle proof \rangle$

lemma *point-in-vector-or-complement*:
assumes *point p*
and *is-vector v*
shows $p \leq v \vee p \leq -v$
$\langle proof \rangle$

lemma *point-in-vector-or-complement-iff*:
assumes *point p*
and *is-vector v*
shows $p \leq v \longleftrightarrow \neg(p \leq -v)$
$\langle proof \rangle$

lemma *different-points-consequences*:
assumes *point p*
and *point q*
and $p \neq q$
shows $p^T;-q=1$

and $-q^T;p=1$
and $-(p^T;-q)=0$
and $-(-q^T;p)=0$
⟨*proof*⟩

Some (more) results about singletons

lemma *singleton-pq*:
assumes *point* p
and *point* q
shows *singleton* $(p;q^T)$
⟨*proof*⟩

lemma *singleton-equal-aux*:
assumes *singleton* p
and *singleton* q
and $q \leq p$
shows $p \leq q;1$
⟨*proof*⟩

lemma *singleton-equal*:
assumes *singleton* p
and *singleton* q
and $q \leq p$
shows $q=p$
⟨*proof*⟩

lemma *singleton-nonsplit*:
assumes *singleton* p
and $x \leq p$
shows $x=0 \vee x=p$
⟨*proof*⟩

lemma *singleton-nonzero*:
assumes *singleton* p
shows $p \neq 0$
⟨*proof*⟩

lemma *singleton-sum*:
assumes *singleton* p
shows $p \leq x+y \longleftrightarrow (p \leq x \vee p \leq y)$
⟨*proof*⟩

lemma *singleton-iff*:
singleton $x \longleftrightarrow x \neq 0 \wedge x^T;1;x + x;1;x^T \leq 1'$
⟨*proof*⟩

lemma *singleton-not-atom-in-relation-algebra-tarski*:
assumes $p \neq 0$
and $\forall x . x \leq p \longrightarrow x=0 \vee x=p$

shows *singleton p*
nitpick [*expect=genuine*] ⟨*proof*⟩

end

1.2 Relation algebras satisfying the point axiom

class *relation-algebra-point* = *relation-algebra* +
assumes *point-axiom*: $x \neq 0 \longrightarrow (\exists y\ z\ .\ point\ y \wedge point\ z \wedge y;z^T \leq x)$
begin

Some (more) results about points

lemma *point-exists*:
$\exists x\ .\ point\ x$
⟨*proof*⟩

lemma *point-below-vector*:
assumes *is-vector v*
and $v \neq 0$
shows $\exists x\ .\ point\ x \wedge x \leq v$
⟨*proof*⟩

end

class *relation-algebra-tarski-point* = *relation-algebra-tarski* +
relation-algebra-point
begin

lemma *atom-is-singleton*:
assumes $p \neq 0$
and $\forall x\ .\ x \leq p \longrightarrow x=0 \vee x=p$
shows *singleton p*
⟨*proof*⟩

lemma *singleton-iff-atom*:
$singleton\ p \longleftrightarrow p \neq 0 \wedge (\forall x\ .\ x \leq p \longrightarrow x=0 \vee x=p)$
⟨*proof*⟩

lemma *maddux-tarski*:
assumes $x \neq 0$
shows $\exists y\ .\ y \neq 0 \wedge y \leq x \wedge is\text{-}p\text{-}fun\ y$
⟨*proof*⟩

Intermediate Point Theorem [10, Proposition 2.4.8]

lemma *intermediate-point-theorem*:
assumes *point p*
and *point r*
shows $p \leq x;y;r \longleftrightarrow (\exists q\ .\ point\ q \wedge p \leq x;q \wedge q \leq y;r)$
⟨*proof*⟩

end

context *relation-algebra*
begin

lemma *unfoldl-inductl-implies-unfoldr*:
assumes $\bigwedge x.\ 1' + x;(rtc\ x) \leq rtc\ x$
and $\bigwedge x\ y\ z.\ x{+}y;z \leq z \Longrightarrow rtc(y);x \leq z$
shows $1' + rtc(x);x \leq rtc\ x$
⟨*proof*⟩

lemma *star-transpose-swap*:
assumes $\bigwedge x.\ 1' + x;(rtc\ x) \leq rtc\ x$
and $\bigwedge x\ y\ z.\ x{+}y;z \leq z \Longrightarrow rtc(y);x \leq z$
shows $rtc(x^T) = (rtc\ x)^T$
⟨*proof*⟩

lemma *unfoldl-inductl-implies-inductr*:
assumes $\bigwedge x.\ 1' + x;(rtc\ x) \leq rtc\ x$
and $\bigwedge x\ y\ z.\ x{+}y;z \leq z \Longrightarrow rtc(y);x \leq z$
shows $x{+}z;y \leq z \Longrightarrow x;rtc(y) \leq z$
⟨*proof*⟩

end

context *relation-algebra-rtc*
begin

abbreviation *tc* (‹(-$^+$)› [*101*] *100*) **where** $tc\ x \equiv x;x^\star$

abbreviation *is-acyclic*
where $is\text{-}acyclic\ x \equiv x^+ \leq -1'$

General theorems

lemma *star-denest-10*:
assumes $x;y{=}0$
shows $(x{+}y)^\star = y;y^\star;x^\star{+}x^\star$
⟨*proof*⟩

lemma *star-star-plus*:
$x^\star + y^\star = x^+ + y^\star$
⟨*proof*⟩

The following two lemmas are from [6].

lemma *cancel-separate*:
assumes $x\ ;\ y \leq 1'$
shows $x^\star\ ;\ y^\star \leq x^\star + y^\star$

⟨*proof*⟩

lemma *cancel-separate-inj-converse*:
assumes *is-inj* x
shows $x^\star ; x^{T\star} = x^\star + x^{T\star}$
⟨*proof*⟩

lemma *cancel-separate-p-fun-converse*:
assumes *is-p-fun* x
shows $x^{T\star} ; x^\star = x^\star + x^{T\star}$
⟨*proof*⟩

lemma *cancel-separate-converse-idempotent*:
assumes *is-inj* x
and *is-p-fun* x
shows $(x^\star + x^{T\star});(x^\star + x^{T\star}) = x^\star + x^{T\star}$
⟨*proof*⟩

lemma *triple-star*:
assumes *is-inj* x
and *is-p-fun* x
shows $x^\star;x^{T\star};x^\star = x^\star + x^{T\star}$
⟨*proof*⟩

lemma *inj-xxts*:
assumes *is-inj* x
shows $x;x^{T\star} \leq x^\star + x^{T\star}$
⟨*proof*⟩

lemma *plus-top*:
$x^+;1 = x;1$
⟨*proof*⟩

lemma *top-plus*:
$1;x^+ = 1;x$
⟨*proof*⟩

lemma *plus-conv*:
$(x^+)^T = x^{T+}$
⟨*proof*⟩

lemma *inj-implies-step-forwards-backwards*:
assumes *is-inj* x
shows $x^\star;(x^+ \cdot 1');1 \leq x^T;1$
⟨*proof*⟩

Acyclic relations

The following result is from [4].

lemma *acyclic-inv*:

assumes *is-acyclic* t
and *is-vector* v
and $e \leq v;-v^T$
and $t \leq v;v^T$
shows *is-acyclic* $(t + e)$
⟨*proof*⟩

lemma *acyclic-single-step*:
assumes *is-acyclic* x
shows $x \leq -1'$
⟨*proof*⟩

lemma *acyclic-reachable-points*:
assumes *is-point* p
and *is-point* q
and $p \leq x;q$
and *is-acyclic* x
shows $p \neq q$
⟨*proof*⟩

lemma *acyclic-trans*:
assumes *is-acyclic* x
shows $x \leq -(x^{T+})$
⟨*proof*⟩

lemma *acyclic-trans'*:
assumes *is-acyclic* x
shows $x^\star \leq -(x^{T+})$
⟨*proof*⟩

Regressively finite

lemma *regressively-finite-acyclic*:
assumes *regressively-finite* x
shows *is-acyclic* x
⟨*proof*⟩

notation *power* (**infixr** ‹↑› *80*)

lemma *power-suc-below-plus*:
$x \uparrow Suc\ n \leq x^+$
⟨*proof*⟩

end

class *relation-algebra-rtc-tarski* = *relation-algebra-rtc* + *relation-algebra-tarski*
begin

lemma *point-loop-not-acyclic*:
assumes *is-point* p

and $p \leq x \uparrow \mathit{Suc}\ n\ ;\ p$
shows $\neg$ *is-acyclic* x
⟨*proof*⟩

end

class *relation-algebra-rtc-point* = *relation-algebra-rtc* + *relation-algebra-point*

class *relation-algebra-rtc-tarski-point* = *relation-algebra-rtc-tarski* + *relation-algebra-rtc-point* +
relation-algebra-tarski-point

Finite graphs: the axiom says the algebra has finitely many elements. This means the relations have a finite base set.

class *relation-algebra-rtc-tarski-point-finite* = *relation-algebra-rtc-tarski-point* + *finite*
begin

For a finite acyclic relation, the powers eventually vanish.

lemma *acyclic-power-vanishes*:
assumes *is-acyclic* x
shows $\exists n\ .\ x \uparrow \mathit{Suc}\ n = \mathit{0}$
⟨*proof*⟩

Hence finite acyclic relations are regressively finite.

lemma *acyclic-regressively-finite*:
assumes *is-acyclic* x
shows *regressively-finite* x
⟨*proof*⟩

lemma *acyclic-is-regressively-finite*:
is-acyclic $x \longleftrightarrow$ *regressively-finite* x
⟨*proof*⟩

end

end

2 Relational Characterisation of Paths

This theory provides the relation-algebraic characterisations of paths, as defined in Sections 3–5 of [2].

theory *Paths*

imports *More-Relation-Algebra*

begin

context *relation-algebra-tarski*
begin

lemma *path-concat-aux-0*:
  assumes *is-vector* v
    and $v \neq 0$
    and $w;v^T \leq x$
    and $v;z \leq y$
  shows $w;1;z \leq x;y$
⟨*proof*⟩

end

2.1 Consequences without the Tarski rule

context *relation-algebra-rtc*
begin

Definitions for path classifications

abbreviation *connected*
  where *connected* $x \equiv x;1;x \leq x^\star + x^{T\star}$

abbreviation *many-strongly-connected*
  where *many-strongly-connected* $x \equiv x^\star = x^{T\star}$

abbreviation *one-strongly-connected*
  where *one-strongly-connected* $x \equiv x^T;1;x^T \leq x^\star$

definition *path*
  where *path* $x \equiv$ *connected* $x \wedge$ *is-p-fun* $x \wedge$ *is-inj* x

abbreviation *cycle*
  where *cycle* $x \equiv$ *path* $x \wedge$ *many-strongly-connected* x

abbreviation *start-points*
  where *start-points* $x \equiv x;1 \cdot -(x^T;1)$

abbreviation *end-points*
  where *end-points* $x \equiv x^T;1 \cdot -(x;1)$

abbreviation *no-start-points*
  where *no-start-points* $x \equiv x;1 \leq x^T;1$

abbreviation *no-end-points*
  where *no-end-points* $x \equiv x^T;1 \leq x;1$

abbreviation *no-start-end-points*
  where *no-start-end-points* $x \equiv x;1 = x^T;1$

abbreviation *has-start-points*

where *has-start-points* $x \equiv 1 = -(1;x);x;1$

abbreviation *has-end-points*
where *has-end-points* $x \equiv 1 = 1;x;-(x;1)$

abbreviation *has-start-end-points*
where *has-start-end-points* $x \equiv 1 = -(1;x);x;1 \cdot 1;x;-(x;1)$

abbreviation *backward-terminating*
where *backward-terminating* $x \equiv x \leq -(1;x);x;1$

abbreviation *forward-terminating*
where *forward-terminating* $x \equiv x \leq 1;x;-(x;1)$

abbreviation *terminating*
where *terminating* $x \equiv x \leq -(1;x);x;1 \cdot 1;x;-(x;1)$

abbreviation *backward-finite*
where *backward-finite* $x \equiv x \leq x^{T\star} + -(1;x);x;1$

abbreviation *forward-finite*
where *forward-finite* $x \equiv x \leq x^{T\star} + 1;x;-(x;1)$

abbreviation *finite*
where *finite* $x \equiv x \leq x^{T\star} + (-(1;x);x;1 \cdot 1;x;-(x;1))$

abbreviation *no-start-points-path*
where *no-start-points-path* $x \equiv$ *path* $x \wedge$ *no-start-points* x

abbreviation *no-end-points-path*
where *no-end-points-path* $x \equiv$ *path* $x \wedge$ *no-end-points* x

abbreviation *no-start-end-points-path*
where *no-start-end-points-path* $x \equiv$ *path* $x \wedge$ *no-start-end-points* x

abbreviation *has-start-points-path*
where *has-start-points-path* $x \equiv$ *path* $x \wedge$ *has-start-points* x

abbreviation *has-end-points-path*
where *has-end-points-path* $x \equiv$ *path* $x \wedge$ *has-end-points* x

abbreviation *has-start-end-points-path*
where *has-start-end-points-path* $x \equiv$ *path* $x \wedge$ *has-start-end-points* x

abbreviation *backward-terminating-path*
where *backward-terminating-path* $x \equiv$ *path* $x \wedge$ *backward-terminating* x

abbreviation *forward-terminating-path*
where *forward-terminating-path* $x \equiv$ *path* $x \wedge$ *forward-terminating* x

abbreviation *terminating-path*
  where *terminating-path* $x \equiv$ *path* $x \wedge$ *terminating* x

abbreviation *backward-finite-path*
  where *backward-finite-path* $x \equiv$ *path* $x \wedge$ *backward-finite* x

abbreviation *forward-finite-path*
  where *forward-finite-path* $x \equiv$ *path* $x \wedge$ *forward-finite* x

abbreviation *finite-path*
  where *finite-path* $x \equiv$ *path* $x \wedge$ *finite* x

General properties

lemma *reachability-from-z-in-y*:
  assumes $x \leq y^\star;z$
    and $x \cdot z = 0$
  shows $x \leq y^+;z$
⟨*proof*⟩

lemma *reachable-imp*:
  assumes *point* p
    and *point* q
    and $p^\star;q \leq p^{T\star};p$
  shows $p \leq p^\star;q$
⟨*proof*⟩

Basic equivalences

lemma *no-start-end-points-iff*:
  no-start-end-points $x \longleftrightarrow$ *no-start-points* $x \wedge$ *no-end-points* x
⟨*proof*⟩

lemma *has-start-end-points-iff*:
  has-start-end-points $x \longleftrightarrow$ *has-start-points* $x \wedge$ *has-end-points* x
⟨*proof*⟩

lemma *terminating-iff*:
  terminating $x \longleftrightarrow$ *backward-terminating* $x \wedge$ *forward-terminating* x
⟨*proof*⟩

lemma *finite-iff*:
  finite $x \longleftrightarrow$ *backward-finite* $x \wedge$ *forward-finite* x
⟨*proof*⟩

lemma *no-start-end-points-path-iff*:
  no-start-end-points-path $x \longleftrightarrow$ *no-start-points-path* $x \wedge$ *no-end-points-path* x
⟨*proof*⟩

lemma *has-start-end-points-path-iff*:

has-start-end-points-path $x \longleftrightarrow$ *has-start-points-path* $x \wedge$ *has-end-points-path* x
⟨*proof*⟩

lemma *terminating-path-iff*:
terminating-path $x \longleftrightarrow$ *backward-terminating-path* $x \wedge$ *forward-terminating-path* x
⟨*proof*⟩

lemma *finite-path-iff*:
finite-path $x \longleftrightarrow$ *backward-finite-path* $x \wedge$ *forward-finite-path* x
⟨*proof*⟩

Closure under converse

lemma *connected-conv*:
connected $x \longleftrightarrow$ *connected* (x^T)
⟨*proof*⟩

lemma *conv-many-strongly-connected*:
many-strongly-connected $x \longleftrightarrow$ *many-strongly-connected* (x^T)
⟨*proof*⟩

lemma *conv-one-strongly-connected*:
one-strongly-connected $x \longleftrightarrow$ *one-strongly-connected* (x^T)
⟨*proof*⟩

lemma *conv-path*:
path $x \longleftrightarrow$ *path* (x^T)
⟨*proof*⟩

lemma *conv-cycle*:
cycle $x \longleftrightarrow$ *cycle* (x^T)
⟨*proof*⟩

lemma *conv-no-start-points*:
no-start-points $x \longleftrightarrow$ *no-end-points* (x^T)
⟨*proof*⟩

lemma *conv-no-start-end-points*:
no-start-end-points $x \longleftrightarrow$ *no-start-end-points* (x^T)
⟨*proof*⟩

lemma *conv-has-start-points*:
has-start-points $x \longleftrightarrow$ *has-end-points* (x^T)
⟨*proof*⟩

lemma *conv-has-start-end-points*:
has-start-end-points $x \longleftrightarrow$ *has-start-end-points* (x^T)
⟨*proof*⟩

lemma *conv-backward-terminating*:
backward-terminating $x \longleftrightarrow$ *forward-terminating* (x^T)
⟨*proof*⟩

lemma *conv-terminating*:
terminating $x \longleftrightarrow$ *terminating* (x^T)
⟨*proof*⟩

lemma *conv-backward-finite*:
backward-finite $x \longleftrightarrow$ *forward-finite* (x^T)
⟨*proof*⟩

lemma *conv-finite*:
finite $x \longleftrightarrow$ *finite* (x^T)
⟨*proof*⟩

lemma *conv-no-start-points-path*:
no-start-points-path $x \longleftrightarrow$ *no-end-points-path* (x^T)
⟨*proof*⟩

lemma *conv-no-start-end-points-path*:
no-start-end-points-path $x \longleftrightarrow$ *no-start-end-points-path* (x^T)
⟨*proof*⟩

lemma *conv-has-start-points-path*:
has-start-points-path $x \longleftrightarrow$ *has-end-points-path* (x^T)
⟨*proof*⟩

lemma *conv-has-start-end-points-path*:
has-start-end-points-path $x \longleftrightarrow$ *has-start-end-points-path* (x^T)
⟨*proof*⟩

lemma *conv-backward-terminating-path*:
backward-terminating-path $x \longleftrightarrow$ *forward-terminating-path* (x^T)
⟨*proof*⟩

lemma *conv-terminating-path*:
terminating-path $x \longleftrightarrow$ *terminating-path* (x^T)
⟨*proof*⟩

lemma *conv-backward-finite-path*:
backward-finite-path $x \longleftrightarrow$ *forward-finite-path* (x^T)
⟨*proof*⟩

lemma *conv-finite-path*:
finite-path $x \longleftrightarrow$ *finite-path* (x^T)
⟨*proof*⟩

Equivalences for *connected*

lemma *connected-iff2*:
assumes *is-inj* x
and *is-p-fun* x
shows *connected* $x \longleftrightarrow x;1;x^T \leq x^\star + x^{T\star}$
⟨*proof*⟩

lemma *connected-iff3*:
assumes *is-inj* x
and *is-p-fun* x
shows *connected* $x \longleftrightarrow x^T;1;x \leq x^\star + x^{T\star}$
⟨*proof*⟩

lemma *connected-iff4*:
connected $x \longleftrightarrow x^T;1;x^T \leq x^\star + x^{T\star}$
⟨*proof*⟩

lemma *connected-iff5*:
connected $x \longleftrightarrow x^+;1;x^+ \leq x^\star + x^{T\star}$
⟨*proof*⟩

lemma *connected-iff6*:
assumes *is-inj* x
and *is-p-fun* x
shows *connected* $x \longleftrightarrow x^+;1;(x^+)^T \leq x^\star + x^{T\star}$
⟨*proof*⟩

lemma *connected-iff7*:
assumes *is-inj* x
and *is-p-fun* x
shows *connected* $x \longleftrightarrow (x^+)^T;1;x^+ \leq x^\star + x^{T\star}$
⟨*proof*⟩

lemma *connected-iff8*:
connected $x \longleftrightarrow (x^+)^T;1;(x^+)^T \leq x^\star + x^{T\star}$
⟨*proof*⟩

Equivalences and implications for *many-strongly-connected*

lemma *many-strongly-connected-iff-1*:
many-strongly-connected $x \longleftrightarrow x^T \leq x^\star$
⟨*proof*⟩

lemma *many-strongly-connected-iff-2*:
many-strongly-connected $x \longleftrightarrow x^T \leq x^+$
⟨*proof*⟩

lemma *many-strongly-connected-iff-3*:
many-strongly-connected $x \longleftrightarrow x \leq x^{T\star}$
⟨*proof*⟩

lemma *many-strongly-connected-iff-4*:
many-strongly-connected $x \longleftrightarrow x \leq x^{T+}$
⟨*proof*⟩

lemma *many-strongly-connected-iff-5*:
many-strongly-connected $x \longleftrightarrow x^\star;x^T \leq x^+$
⟨*proof*⟩

lemma *many-strongly-connected-iff-6*:
many-strongly-connected $x \longleftrightarrow x^T;x^\star \leq x^+$
⟨*proof*⟩

lemma *many-strongly-connected-iff-7*:
many-strongly-connected $x \longleftrightarrow x^{T+} = x^+$
⟨*proof*⟩

lemma *many-strongly-connected-iff-5-eq*:
many-strongly-connected $x \longleftrightarrow x^\star;x^T = x^+$
⟨*proof*⟩

lemma *many-strongly-connected-iff-6-eq*:
many-strongly-connected $x \longleftrightarrow x^T;x^\star = x^+$
⟨*proof*⟩

lemma *many-strongly-connected-implies-no-start-end-points*:
assumes *many-strongly-connected* x
shows *no-start-end-points* x
⟨*proof*⟩

lemma *many-strongly-connected-implies-8*:
assumes *many-strongly-connected* x
shows $x;x^T \leq x^+$
⟨*proof*⟩

lemma *many-strongly-connected-implies-9*:
assumes *many-strongly-connected* x
shows $x^T;x \leq x^+$
⟨*proof*⟩

lemma *many-strongly-connected-implies-10*:
assumes *many-strongly-connected* x
shows $x;x^T;x^\star \leq x^+$
⟨*proof*⟩

lemma *many-strongly-connected-implies-10-eq*:
assumes *many-strongly-connected* x
shows $x;x^T;x^\star = x^+$
⟨*proof*⟩

lemma *many-strongly-connected-implies-11*:
assumes *many-strongly-connected* x
shows $x^\star;x^T;x \leq x^+$
⟨*proof*⟩

lemma *many-strongly-connected-implies-11-eq*:
assumes *many-strongly-connected* x
shows $x^\star;x^T;x = x^+$
⟨*proof*⟩

lemma *many-strongly-connected-implies-12*:
assumes *many-strongly-connected* x
shows $x^\star;x;x^T \leq x^+$
⟨*proof*⟩

lemma *many-strongly-connected-implies-12-eq*:
assumes *many-strongly-connected* x
shows $x^\star;x;x^T = x^+$
⟨*proof*⟩

lemma *many-strongly-connected-implies-13*:
assumes *many-strongly-connected* x
shows $x^T;x;x^\star \leq x^+$
⟨*proof*⟩

lemma *many-strongly-connected-implies-13-eq*:
assumes *many-strongly-connected* x
shows $x^T;x;x^\star = x^+$
⟨*proof*⟩

lemma *many-strongly-connected-iff-8*:
assumes *is-p-fun* x
shows *many-strongly-connected* $x \longleftrightarrow x;x^T \leq x^+$
⟨*proof*⟩

lemma *many-strongly-connected-iff-9*:
assumes *is-inj* x
shows *many-strongly-connected* $x \longleftrightarrow x^T;x \leq x^+$
⟨*proof*⟩

lemma *many-strongly-connected-iff-10*:
assumes *is-p-fun* x
shows *many-strongly-connected* $x \longleftrightarrow x;x^T;x^\star \leq x^+$
⟨*proof*⟩

lemma *many-strongly-connected-iff-10-eq*:
assumes *is-p-fun* x
shows *many-strongly-connected* $x \longleftrightarrow x;x^T;x^\star = x^+$
⟨*proof*⟩

lemma *many-strongly-connected-iff-11*:
  assumes *is-inj* x
    shows *many-strongly-connected* $x \longleftrightarrow x^\star;x^T;x \leq x^+$
⟨*proof*⟩

lemma *many-strongly-connected-iff-11-eq*:
  assumes *is-inj* x
    shows *many-strongly-connected* $x \longleftrightarrow x^\star;x^T;x = x^+$
⟨*proof*⟩

lemma *many-strongly-connected-iff-12*:
  assumes *is-p-fun* x
    shows *many-strongly-connected* $x \longleftrightarrow x^\star;x;x^T \leq x^+$
⟨*proof*⟩

lemma *many-strongly-connected-iff-12-eq*:
  assumes *is-p-fun* x
    shows *many-strongly-connected* $x \longleftrightarrow x^\star;x;x^T = x^+$
⟨*proof*⟩

lemma *many-strongly-connected-iff-13*:
  assumes *is-inj* x
    shows *many-strongly-connected* $x \longleftrightarrow x^T;x;x^\star \leq x^+$
⟨*proof*⟩

lemma *many-strongly-connected-iff-13-eq*:
  assumes *is-inj* x
    shows *many-strongly-connected* $x \longleftrightarrow x^T;x;x^\star = x^+$
⟨*proof*⟩

Equivalences and implications for *one-strongly-connected*

lemma *one-strongly-connected-iff*:
  one-strongly-connected $x \longleftrightarrow$ *connected* $x \wedge$ *many-strongly-connected* x
  ⟨*proof*⟩

lemma *one-strongly-connected-iff-1*:
  one-strongly-connected $x \longleftrightarrow x^T;1;x^T \leq x^+$
⟨*proof*⟩

lemma *one-strongly-connected-iff-1-eq*:
  one-strongly-connected $x \longleftrightarrow x^T;1;x^T = x^+$
  ⟨*proof*⟩

lemma *one-strongly-connected-iff-2*:
  one-strongly-connected $x \longleftrightarrow x;1;x \leq x^{T\star}$
⟨*proof*⟩

lemma *one-strongly-connected-iff-3*:

one-strongly-connected $x \longleftrightarrow x;1;x \leq x^{T+}$
⟨*proof*⟩

lemma *one-strongly-connected-iff-3-eq*:
one-strongly-connected $x \longleftrightarrow x;1;x = x^{T+}$
⟨*proof*⟩

lemma *one-strongly-connected-iff-4-eq*:
one-strongly-connected $x \longleftrightarrow x^T;1;x = x^+$
⟨*proof*⟩

lemma *one-strongly-connected-iff-5-eq*:
one-strongly-connected $x \longleftrightarrow x;1;x^T = x^+$
⟨*proof*⟩

lemma *one-strongly-connected-iff-6-aux*:
$x;x^+ \leq x;1;x$
⟨*proof*⟩

lemma *one-strongly-connected-implies-6-eq*:
assumes *one-strongly-connected* x
shows $x;1;x = x;x^+$
⟨*proof*⟩

lemma *one-strongly-connected-iff-7-aux*:
$x^+ \leq x;1;x$
⟨*proof*⟩

lemma *one-strongly-connected-implies-7-eq*:
assumes *one-strongly-connected* x
shows $x;1;x = x^+$
⟨*proof*⟩

lemma *one-strongly-connected-implies-8*:
assumes *one-strongly-connected* x
shows $x;1;x \leq x^\star$
⟨*proof*⟩

lemma *one-strongly-connected-iff-4*:
assumes *is-inj* x
shows *one-strongly-connected* $x \longleftrightarrow x^T;1;x \leq x^+$
⟨*proof*⟩

lemma *one-strongly-connected-iff-5*:
assumes *is-p-fun* x
shows *one-strongly-connected* $x \longleftrightarrow x;1;x^T \leq x^+$
⟨*proof*⟩

lemma *one-strongly-connected-iff-6*:

assumes *is-p-fun* x
and *is-inj* x
shows *one-strongly-connected* $x \longleftrightarrow x;1;x \leq x;x^+$
⟨*proof*⟩

lemma *one-strongly-connected-iff-6-eq*:
assumes *is-p-fun* x
and *is-inj* x
shows *one-strongly-connected* $x \longleftrightarrow x;1;x = x;x^+$
⟨*proof*⟩

Start points and end points

lemma *start-end-implies-terminating*:
assumes *has-start-points* x
and *has-end-points* x
shows *terminating* x
⟨*proof*⟩

lemma *start-points-end-points-conv*:
start-points x = *end-points* (x^T)
⟨*proof*⟩

lemma *start-point-at-most-one*:
assumes *path* x
shows *is-inj* (*start-points* x)
⟨*proof*⟩

lemma *start-point-zero-point*:
assumes *path* x
shows *start-points* x = 0 ∨ *is-point* (*start-points* x)
⟨*proof*⟩

lemma *start-point-iff1*:
assumes *path* x
shows *is-point* (*start-points* x) ⟷ ¬(*no-start-points* x)
⟨*proof*⟩

lemma *end-point-at-most-one*:
assumes *path* x
shows *is-inj* (*end-points* x)
⟨*proof*⟩

lemma *end-point-zero-point*:
assumes *path* x
shows *end-points* x = 0 ∨ *is-point* (*end-points* x)
⟨*proof*⟩

lemma *end-point-iff1*:
assumes *path* x

shows *is-point* (*end-points* x) $\longleftrightarrow$ $\neg$(*no-end-points* x)
⟨*proof*⟩

lemma *predecessor-point'*:
assumes *path* x
and *point* s
and *point* e
and $e;s^T \leq x$
shows $x;s = e$
⟨*proof*⟩

lemma *predecessor-point*:
assumes *path* x
and *point* s
and *point* e
and $e;s^T \leq x$
shows *point*$(x;s)$
⟨*proof*⟩

lemma *points-of-path-iff*:
shows $(x + x^T);1 = x^T;1 +$ *start-points*(x)
and $(x + x^T);1 = x;1 +$ *end-points*(x)
⟨*proof*⟩

Path concatenation preliminaries

lemma *path-concat-aux-1*:
assumes $x;1 \cdot y;1 \cdot y^T;1 = 0$
and *end-points* x = *start-points* y
shows $x;1 \cdot y;1 = 0$
⟨*proof*⟩

lemma *path-concat-aux-2*:
assumes $x;1 \cdot x^T;1 \cdot y^T;1 = 0$
and *end-points* x = *start-points* y
shows $x^T;1 \cdot y^T;1 = 0$
⟨*proof*⟩

lemma *path-concat-aux3-1*:
assumes *path* x
shows $x;1;x^T \leq x^\star + x^{T\star}$
⟨*proof*⟩

lemma *path-concat-aux3-2*:
assumes *path* x
shows $x^T;1;x \leq x^\star + x^{T\star}$
⟨*proof*⟩

lemma *path-concat-aux3-3*:
assumes *path* x

shows $x^T;1;x^T \leq x^\star + x^{T\star}$
⟨*proof*⟩

lemma *path-concat-aux-3*:
assumes *path* x
and $y \leq x^+ + x^{T+}$
and $z \leq x^+ + x^{T+}$
shows $y;1;z \leq x^\star + x^{T\star}$
⟨*proof*⟩

lemma *path-concat-aux-4*:
$x^\star + x^{T\star} \leq x^\star + x^T;1$
⟨*proof*⟩

lemma *path-concat-aux-5*:
assumes *path* x
and $y \leq$ *start-points* x
and $z \leq x + x^T$
shows $y;1;z \leq x^\star$
⟨*proof*⟩

lemma *path-conditions-disjoint-points-iff*:
$x;1 \cdot (x^T;1 + y;1) \cdot y^T;1 = 0 \wedge$ *start-points* $x \cdot$ *end-points* $y = 0 \longleftrightarrow x;1 \cdot y^T;1 = 0$
⟨*proof*⟩

end

2.2 Consequences with the Tarski rule

context *relation-algebra-rtc-tarski*
begin

General theorems

lemma *reachable-implies-predecessor*:
assumes $p \neq q$
and *point* p
and *point* q
and $x^\star;q \leq x^{T\star};p$
shows $x;q \neq 0$
⟨*proof*⟩

lemma *acyclic-imp-one-step-different-points*:
assumes *is-acyclic* x
and *point* p
and *point* q
and $p \leq x;q$
shows $p \leq -q$ **and** $p \neq q$
⟨*proof*⟩

Start points and end points

lemma *start-point-iff2*:
assumes *path* x
shows *is-point* (*start-points* x) $\longleftrightarrow$ *has-start-points* x
⟨*proof*⟩

lemma *end-point-iff2*:
assumes *path* x
shows *is-point* (*end-points* x) $\longleftrightarrow$ *has-end-points* x
⟨*proof*⟩

lemma *edge-is-path*:
assumes *is-point* p
and *is-point* q
shows *path* $(p;q^T)$
⟨*proof*⟩

lemma *edge-start*:
assumes *is-point* p
and *is-point* q
and $p \neq q$
shows *start-points* $(p;q^T) = p$
⟨*proof*⟩

lemma *edge-end*:
assumes *is-point* p
and *is-point* q
and $p \neq q$
shows *end-points* $(p;q^T) = q$
⟨*proof*⟩

lemma *loop-no-start*:
assumes *is-point* p
shows *start-points* $(p;p^T) = 0$
⟨*proof*⟩

lemma *loop-no-end*:
assumes *is-point* p
shows *end-points* $(p;p^T) = 0$
⟨*proof*⟩

lemma *start-point-no-predecessor*:
x;*start-points*$(x) = 0$
⟨*proof*⟩

lemma *end-point-no-successor*:
x^T;*end-points*$(x) = 0$
⟨*proof*⟩

lemma *start-to-end*:
  assumes *path* x
    shows *start-points*(x);*end-points*$(x)^T \leq x^\star$
⟨*proof*⟩

lemma *path-acyclic*:
  assumes *has-start-points-path* x
    shows *is-acyclic* x
⟨*proof*⟩

Equivalences for *terminating*

lemma *backward-terminating-iff1*:
  assumes *path* x
    shows *backward-terminating* $x \longleftrightarrow$ *has-start-points* $x \vee x = 0$
⟨*proof*⟩

lemma *backward-terminating-iff2-aux*:
  assumes *path* x
    shows $x;1 \cdot 1;x^T \cdot -(1;x) \leq x^{T\star}$
⟨*proof*⟩

lemma *backward-terminating-iff2*:
  assumes *path* x
    shows *backward-terminating* $x \longleftrightarrow x \leq x^{T\star};-(x^T;1)$
⟨*proof*⟩

lemma *backward-terminating-iff3-aux*:
  assumes *path* x
    shows $x^T;1 \cdot 1;x^T \cdot -(1;x) \leq x^{T\star}$
⟨*proof*⟩

lemma *backward-terminating-iff3*:
  assumes *path* x
    shows *backward-terminating* $x \longleftrightarrow x^T \leq x^{T\star};-(x^T;1)$
⟨*proof*⟩

lemma *backward-terminating-iff4*:
  assumes *path* x
    shows *backward-terminating* $x \longleftrightarrow x \leq -(1;x);x^\star$
⟨*proof*⟩

lemma *forward-terminating-iff1*:
  assumes *path* x
    shows *forward-terminating* $x \longleftrightarrow$ *has-end-points* $x \vee x = 0$
⟨*proof*⟩

lemma *forward-terminating-iff2*:
  assumes *path* x
    shows *forward-terminating* $x \longleftrightarrow x^T \leq x^\star;-(x;1)$

⟨*proof*⟩

lemma *forward-terminating-iff3*:
assumes *path x*
shows *forward-terminating x* $\longleftrightarrow$ $x \leq x^\star;-(x;1)$
⟨*proof*⟩

lemma *forward-terminating-iff4*:
assumes *path x*
shows *forward-terminating x* $\longleftrightarrow$ $x \leq -(1;x^T);x^{T\star}$
⟨*proof*⟩

lemma *terminating-iff1*:
assumes *path x*
shows *terminating x* $\longleftrightarrow$ *has-start-end-points x* $\vee$ $x = 0$
⟨*proof*⟩

lemma *terminating-iff2*:
assumes *path x*
shows *terminating x* $\longleftrightarrow$ $x \leq x^{T\star};-(x^T;1) \cdot -(1;x^T);x^{T\star}$
⟨*proof*⟩

lemma *terminating-iff3*:
assumes *path x*
shows *terminating x* $\longleftrightarrow$ $x \leq x^\star;-(x;1) \cdot -(1;x);x^\star$
⟨*proof*⟩

lemma *backward-terminating-path-irreflexive*:
assumes *backward-terminating-path x*
shows $x \leq -1'$
⟨*proof*⟩

lemma *forward-terminating-path-end-points-1*:
assumes *forward-terminating-path x*
shows $x \leq x^+;$*end-points x*
⟨*proof*⟩

lemma *forward-terminating-path-end-points-2*:
assumes *forward-terminating-path x*
shows $x^T \leq x^\star;$*end-points x*
⟨*proof*⟩

lemma *forward-terminating-path-end-points-3*:
assumes *forward-terminating-path x*
shows *start-points x* $\leq x^+;$*end-points x*
⟨*proof*⟩

lemma *backward-terminating-path-start-points-1*:
assumes *backward-terminating-path x*

shows $x^T \leq x^{T+}$;*start-points* x
⟨*proof*⟩

lemma *backward-terminating-path-start-points-2*:
assumes *backward-terminating-path* x
shows $x \leq x^{T\star}$;*start-points* x
⟨*proof*⟩

lemma *backward-terminating-path-start-points-3*:
assumes *backward-terminating-path* x
shows *end-points* $x \leq x^{T+}$;*start-points* x
⟨*proof*⟩

lemma *path-aux1a*:
assumes *forward-terminating-path* x
shows $x \neq 0 \longleftrightarrow$ *end-points* $x \neq 0$
⟨*proof*⟩

lemma *path-aux1b*:
assumes *backward-terminating-path* y
shows $y \neq 0 \longleftrightarrow$ *start-points* $y \neq 0$
⟨*proof*⟩

lemma *path-aux1*:
assumes *forward-terminating-path* x
and *backward-terminating-path* y
shows $x \neq 0 \vee y \neq 0 \longleftrightarrow$ *end-points* $x \neq 0 \vee$ *start-points* $y \neq 0$
⟨*proof*⟩

Equivalences for *finite*

lemma *backward-finite-iff-msc*:
backward-finite $x \longleftrightarrow$ *many-strongly-connected* $x \vee$ *backward-terminating* x
⟨*proof*⟩

lemma *forward-finite-iff-msc*:
forward-finite $x \longleftrightarrow$ *many-strongly-connected* $x \vee$ *forward-terminating* x
⟨*proof*⟩

lemma *finite-iff-msc*:
finite $x \longleftrightarrow$ *many-strongly-connected* $x \vee$ *terminating* x
⟨*proof*⟩

Path concatenation

lemma *path-concatenation*:
assumes *forward-terminating-path* x
and *backward-terminating-path* y

and *end-points* x = *start-points* y
and $x;1 \cdot (x^T;1 + y;1) \cdot y^T;1 = 0$
shows *path* $(x+y)$
⟨*proof*⟩

lemma *path-concatenation-with-edge*:
assumes $x \neq 0$
and *forward-terminating-path* x
and *is-point* q
and $q \leq -(1;x)$
shows *path* $(x+($*end-points* $x);q^T)$
⟨*proof*⟩

lemma *path-concatenation-cycle-free*:
assumes *forward-terminating-path* x
and *backward-terminating-path* y
and *end-points* x = *start-points* y
and $x;1 \cdot y^T;1 = 0$
shows *path* $(x+y)$
⟨*proof*⟩

lemma *path-concatenation-start-points-approx*:
assumes *end-points* x = *start-points* y
shows *start-points* $(x+y) \leq$ *start-points* x
⟨*proof*⟩

lemma *path-concatenation-end-points-approx*:
assumes *end-points* x = *start-points* y
shows *end-points* $(x+y) \leq$ *end-points* y
⟨*proof*⟩

lemma *path-concatenation-start-points*:
assumes *end-points* x = *start-points* y
and $x;1 \cdot y^T;1 = 0$
shows *start-points* $(x+y)$ = *start-points* x
⟨*proof*⟩

lemma *path-concatenation-end-points*:
assumes *end-points* x = *start-points* y
and $x;1 \cdot y^T;1 = 0$
shows *end-points* $(x+y)$ = *end-points* y
⟨*proof*⟩

lemma *path-concatenation-cycle-free-complete*:
assumes *forward-terminating-path* x
and *backward-terminating-path* y
and *end-points* x = *start-points* y
and $x;1 \cdot y^T;1 = 0$
shows *path* $(x+y) \wedge$ *start-points* $(x+y)$ = *start-points* $x \wedge$ *end-points* $(x+y)$

$= \textit{end-points } y$
⟨*proof*⟩

Path restriction (path from a given point)

lemma *reachable-points-iff*:
assumes *point p*
shows $(x^{T\star};p \cdot x) = (x^{T\star};p \cdot 1');x$
⟨*proof*⟩

lemma *path-from-given-point*:
assumes *path x*
and *point p*
shows $path(x^{T\star};p \cdot x)$
and $start\text{-}points(x^{T\star};p \cdot x) \leq p$
and $end\text{-}points(x^{T\star};p \cdot x) \leq end\text{-}points(x)$
⟨*proof*⟩

lemma *path-from-given-point'*:
assumes *has-start-points-path x*
and *point p*
and $p \leq x;1$
shows $path(x^{T\star};p \cdot x)$
and $start\text{-}points(x^{T\star};p \cdot x) = p$
and $end\text{-}points(x^{T\star};p \cdot x) = end\text{-}points(x)$
⟨*proof*⟩

Cycles

lemma *selfloop-is-cycle*:
assumes *is-point x*
shows $cycle\ (x;x^T)$
⟨*proof*⟩

lemma *start-point-no-cycle*:
assumes *has-start-points-path x*
shows $\neg$ *cycle x*
⟨*proof*⟩

lemma *end-point-no-cycle*:
assumes *has-end-points-path x*
shows $\neg$ *cycle x*
⟨*proof*⟩

lemma *cycle-no-points*:
assumes *cycle x*
shows $start\text{-}points\ x = 0$
and $end\text{-}points\ x = 0$
⟨*proof*⟩

Path concatenation to cycle

lemma *path-path-equals-cycle-aux*:
  assumes *has-start-end-points-path* x
    and *has-start-end-points-path* y
    and *start-points* x = *end-points* y
    and *end-points* x = *start-points* y
shows $x \leq (x+y)^{T\star}$
⟨*proof*⟩

lemma *path-path-equals-cycle*:
  assumes *has-start-end-points-path* x
    and *has-start-end-points-path* y
    and *start-points* x = *end-points* y
    and *end-points* x = *start-points* y
    and $x;1 \cdot (x^T;1 + y;1) \cdot y^T;1 = 0$
  shows *cycle*$(x + y)$
⟨*proof*⟩

lemma *path-edge-equals-cycle*:
  assumes *has-start-end-points-path* x
  shows *cycle*(x + *end-points*(x);(*start-points* x)^T)
⟨*proof*⟩

Break cycles

lemma *cycle-remove-edge*:
  assumes *cycle* x
    and *point* s
    and *point* e
    and $e;s^T \leq x$
  shows *path*$(x \cdot -(e;s^T))$
    and *start-points* $(x \cdot -(e;s^T)) \leq s$
    and *end-points* $(x \cdot -(e;s^T)) \leq e$
⟨*proof*⟩

lemma *cycle-remove-edge'*:
  assumes *cycle* x
    and *point* s
    and *point* e
    and $s \neq e$
    and $e;s^T \leq x$
  shows *path*$(x \cdot -(e;s^T))$
    and s = *start-points* $(x \cdot -(e;s^T))$
    and e = *end-points* $(x \cdot -(e;s^T))$
⟨*proof*⟩

end

end

3 Relational Characterisation of Rooted Paths

We characterise paths together with a designated root. This is important as often algorithms start with a single vertex, and then build up a path, a tree or another structure. An example is Dijkstra's shortest path algorithm.

theory *Rooted-Paths*

imports *Paths*

begin

context *relation-algebra*
begin

General theorems

lemma *step-has-target*:
  assumes $x;r \neq 0$
    shows $x^T;1 \neq 0$
⟨*proof*⟩

lemma *end-point-char*:
  $x^T;p = 0 \longleftrightarrow p \leq -(x;1)$
⟨*proof*⟩

end

context *relation-algebra-tarski*
begin

General theorems concerning points

lemma *successor-point*:
  assumes *is-inj* x
    and *point* r
    and $x;r \neq 0$
  shows *point* $(x;r)$
⟨*proof*⟩

lemma *no-end-point-char*:
  assumes *point* p
    shows $x^T;p \neq 0 \longleftrightarrow p \leq x;1$
⟨*proof*⟩

lemma *no-end-point-char-converse*:
  assumes *point* p
    shows $x;p \neq 0 \longleftrightarrow p \leq x^T;1$
⟨*proof*⟩

end

3.1 Consequences without the Tarski rule

context *relation-algebra-rtc*
begin

Definitions for path classifications

definition *path-root*
where *path-root* $r\ x \equiv r;x \leq x^\star + x^{T\star} \wedge$ *is-inj* $x \wedge$ *is-p-fun* $x \wedge$ *point* r

abbreviation *connected-root*
where *connected-root* $r\ x \equiv r;x \leq x^+$

definition *backward-finite-path-root*
where *backward-finite-path-root* $r\ x \equiv$ *connected-root* $r\ x \wedge$ *is-inj* $x \wedge$ *is-p-fun* x $\wedge$ *point* r

abbreviation *backward-terminating-path-root*
where *backward-terminating-path-root* $r\ x \equiv$ *backward-finite-path-root* $r\ x \wedge x;r$ $= 0$

abbreviation *cycle-root*
where *cycle-root* $r\ x \equiv r;x \leq x^+ \cdot x^T;1 \wedge$ *is-inj* $x \wedge$ *is-p-fun* $x \wedge$ *point* r

abbreviation *non-empty-cycle-root*
where *non-empty-cycle-root* $r\ x \equiv$ *backward-finite-path-root* $r\ x \wedge r \leq x^T;1$

abbreviation *finite-path-root-end*
where *finite-path-root-end* $r\ x\ e \equiv$ *backward-finite-path-root* $r\ x \wedge$ *point* $e \wedge r \leq$ $x^\star;e$

abbreviation *terminating-path-root-end*
where *terminating-path-root-end* $r\ x\ e \equiv$ *finite-path-root-end* $r\ x\ e \wedge x^T;e = 0$

Equivalent formulations of *connected-root*

lemma *connected-root-iff1*:
assumes *point* r
shows *connected-root* $r\ x \longleftrightarrow 1;x \leq r^T;x^+$
⟨*proof*⟩

lemma *connected-root-iff2*:
assumes *point* r
shows *connected-root* $r\ x \longleftrightarrow x^T;1 \leq x^{T+};r$
⟨*proof*⟩

lemma *connected-root-aux*:
$x^{T+};r \leq x^T;1$
⟨*proof*⟩

lemma *connected-root-iff3*:
assumes *point* r

shows *connected-root r x* $\longleftrightarrow$ $x^T;1 = x^{T+};r$
⟨*proof*⟩

lemma *connected-root-iff4*:
assumes *point r*
shows *connected-root r x* $\longleftrightarrow$ $1;x = r^T;x^+$
⟨*proof*⟩

Consequences of *connected-root*

lemma *has-root-contra*:
assumes *connected-root r x*
and *point r*
and $x^T;r = 0$
shows $x = 0$
⟨*proof*⟩

lemma *has-root*:
assumes *connected-root r x*
and *point r*
and $x \neq 0$
shows $x^T;r \neq 0$
⟨*proof*⟩

lemma *connected-root-move-root*:
assumes *connected-root r x*
and $q \leq x^\star;r$
shows *connected-root q x*
⟨*proof*⟩

lemma *root-cycle-converse*:
assumes *connected-root r x*
and *point r*
and $x;r \neq 0$
shows $x^T;r \neq 0$
⟨*proof*⟩

Rooted paths

lemma *path-iff-aux-1*:
assumes *bijective r*
shows $r;x \leq x^\star + x^{T\star} \longleftrightarrow x \leq r^T;(x^\star + x^{T\star})$
⟨*proof*⟩

lemma *path-iff-aux-2*:
assumes *bijective r*
shows $r;x \leq x^\star + x^{T\star} \longleftrightarrow x^T \leq (x^\star + x^{T\star});r$
⟨*proof*⟩

lemma *path-iff-backward*:
assumes *is-inj x*

and *is-p-fun* x
and *point* r
and $r;x \leq x^\star + x^{T\star}$
shows *connected* x
⟨*proof*⟩

lemma *empty-path-root-end*:
assumes *terminating-path-root-end* $r\ x\ e$
shows $e = r \longleftrightarrow x = 0$
⟨*proof*⟩

lemma *path-root-acyclic*:
assumes *path-root* $r\ x$
and $x;r = 0$
shows *is-acyclic* x
⟨*proof*⟩

Start points and end points

lemma *start-points-in-root-aux*:
assumes *backward-finite-path-root* $r\ x$
shows $x;1 \leq x^{T\star};r$
⟨*proof*⟩

lemma *start-points-in-root*:
assumes *backward-finite-path-root* $r\ x$
shows *start-points* $x \leq r$
⟨*proof*⟩

lemma *start-points-not-zero-contra*:
assumes *connected-root* $r\ x$
and *point* r
and *start-points* $x = 0$
and $x;r = 0$
shows $x = 0$
⟨*proof*⟩

lemma *start-points-not-zero*:
assumes *connected-root* $r\ x$
and *point* r
and $x \neq 0$
and $x;r = 0$
shows *start-points* $x \neq 0$
⟨*proof*⟩

Backwards terminating and backwards finite

lemma *backward-terminating-path-root-aux*:
assumes *backward-terminating-path-root* $r\ x$
shows $x \leq x^{T\star};-(x^T;1)$
⟨*proof*⟩

lemma *backward-finite-path-connected-aux*:
assumes *backward-finite-path-root r x*
shows $x^T;r;x^T \le x^\star + x^{T\star}$
⟨*proof*⟩

lemma *backward-finite-path-connected*:
assumes *backward-finite-path-root r x*
shows *connected x*
⟨*proof*⟩

lemma *backward-finite-path-root-path*:
assumes *backward-finite-path-root r x*
shows *path x*
⟨*proof*⟩

lemma *backward-finite-path-root-path-root*:
assumes *backward-finite-path-root r x*
shows *path-root r x*
⟨*proof*⟩

lemma *zero-backward-terminating-path-root*:
assumes *point r*
shows *backward-terminating-path-root r 0*
⟨*proof*⟩

lemma *backward-finite-path-root-move-root*:
assumes *backward-finite-path-root r x*
and *point q*
and $q \le x^\star;r$
shows *backward-finite-path-root q x*
⟨*proof*⟩

Cycle

lemma *non-empty-cycle-root-var-axioms-1*:
non-empty-cycle-root r x $\longleftrightarrow x^T;1 \le x^{T+};r \land$ *is-inj x* $\land$ *is-p-fun x* $\land$ *point r* $\land$ $r \le x^T;1$
⟨*proof*⟩

lemma *non-empty-cycle-root-loop*:
assumes *non-empty-cycle-root r x*
shows $r \le x^{T+};r$
⟨*proof*⟩

lemma *cycle-root-end-empty*:
assumes *terminating-path-root-end r x e*
and *many-strongly-connected x*
shows $x = 0$
⟨*proof*⟩

lemma *cycle-root-end-empty-var*:
assumes *terminating-path-root-end r x e*
and $x \neq 0$
shows $\neg$ *many-strongly-connected x*
⟨*proof*⟩

Terminating path

lemma *terminating-path-root-end-connected*:
assumes *terminating-path-root-end r x e*
shows $x;1 \leq x^+;e$
⟨*proof*⟩

lemma *terminating-path-root-end-forward-finite*:
assumes *terminating-path-root-end r x e*
shows *backward-finite-path-root e* (x^T)
⟨*proof*⟩

end

3.2 Consequences with the Tarski rule

context *relation-algebra-rtc-tarski*
begin

Some (more) results about points

lemma *point-reachable-converse*:
assumes *is-vector v*
and $v \neq 0$
and *point r*
and $v \leq x^{T+};r$
shows $r \leq x^+;v$
⟨*proof*⟩

Roots

lemma *root-in-start-points*:
assumes *connected-root r x*
and *is-vector r*
and $x \neq 0$
and $x;r = 0$
shows $r \leq$ *start-points x*
⟨*proof*⟩

lemma *root-equals-start-points*:
assumes *backward-terminating-path-root r x*
and $x \neq 0$
shows $r =$ *start-points x*
⟨*proof*⟩

lemma *root-equals-end-points*:
assumes *backward-terminating-path-root* r (x^T)
and $x \neq 0$
shows $r = \textit{end-points}\ x$
⟨*proof*⟩

lemma *root-in-edge-sources*:
assumes *connected-root* r x
and $x \neq 0$
and *is-vector* r
shows $r \leq x;1$
⟨*proof*⟩

Rooted Paths

lemma *non-empty-path-root-iff-aux*:
assumes *path-root* r x
and $x \neq 0$
shows $r \leq (x + x^T);1$
⟨*proof*⟩

Backwards terminating and backwards finite

lemma *backward-terminating-path-root-2*:
assumes *backward-terminating-path-root* r x
shows *backward-terminating* x
⟨*proof*⟩

lemma *backward-terminating-path-root*:
assumes *backward-terminating-path-root* r x
shows *backward-terminating-path* x
⟨*proof*⟩

(Non-empty) Cycle

lemma *cycle-iff*:
assumes *point* r
shows $x;r \neq 0 \longleftrightarrow r \leq x^T;1$
⟨*proof*⟩

lemma *non-empty-cycle-root-iff*:
assumes *connected-root* r x
and *point* r
shows $x;r \neq 0 \longleftrightarrow r \leq x^{T+};r$
⟨*proof*⟩

lemma *backward-finite-path-root-terminating-or-cycle*:
backward-finite-path-root r x $\longleftrightarrow$ *backward-terminating-path-root* r x $\vee$ *non-empty-cycle-root* r x
⟨*proof*⟩

lemma *non-empty-cycle-root-msc*:

assumes *non-empty-cycle-root r x*
shows *many-strongly-connected x*
⟨*proof*⟩

lemma *non-empty-cycle-root-msc-cycle*:
assumes *non-empty-cycle-root r x*
shows *cycle x*
⟨*proof*⟩

lemma *non-empty-cycle-root-non-empty*:
assumes *non-empty-cycle-root r x*
shows $x \neq 0$
⟨*proof*⟩

lemma *non-empty-cycle-root-rtc-symmetric*:
assumes *non-empty-cycle-root r x*
shows $x^\star;r = x^{T\star};r$
⟨*proof*⟩

lemma *non-empty-cycle-root-point-exchange*:
assumes *non-empty-cycle-root r x*
and *point p*
shows $r \leq x^\star;p \longleftrightarrow p \leq x^\star;r$
⟨*proof*⟩

lemma *non-empty-cycle-root-rtc-tc*:
assumes *non-empty-cycle-root r x*
shows $x^\star;r = x^+;r$
⟨*proof*⟩

lemma *non-empty-cycle-root-no-start-end-points*:
assumes *non-empty-cycle-root r x*
shows $x;1 = x^T;1$
⟨*proof*⟩

lemma *non-empty-cycle-root-move-root*:
assumes *non-empty-cycle-root r x*
and *point q*
and $q \leq x^\star;r$
shows *non-empty-cycle-root q x*
⟨*proof*⟩

lemma *non-empty-cycle-root-loop-converse*:
assumes *non-empty-cycle-root r x*
shows $r \leq x^+;r$
⟨*proof*⟩

lemma *non-empty-cycle-root-move-root-same-reachable*:
assumes *non-empty-cycle-root r x*

and *point* q
and $q \le x^\star;r$
shows $x^\star;r = x^\star;q$
⟨*proof*⟩

lemma *non-empty-cycle-root-move-root-same-reachable-2*:
assumes *non-empty-cycle-root* r x
and *point* q
and $q \le x^\star;r$
shows $x^\star;r = x^{T\star};q$
⟨*proof*⟩

lemma *non-empty-cycle-root-move-root-msc*:
assumes *non-empty-cycle-root* r x
shows $x^{T\star};q = x^\star;q$
⟨*proof*⟩

lemma *non-empty-cycle-root-move-root-rtc-tc*:
assumes *non-empty-cycle-root* r x
and *point* q
and $q \le x^\star;r$
shows $x^\star;q = x^+;q$
⟨*proof*⟩

lemma *non-empty-cycle-root-move-root-loop-converse*:
assumes *non-empty-cycle-root* r x
and *point* q
and $q \le x^\star;r$
shows $q \le x^{T+};q$
⟨*proof*⟩

lemma *non-empty-cycle-root-move-root-loop*:
assumes *non-empty-cycle-root* r x
and *point* q
and $q \le x^\star;r$
shows $q \le x^+;q$
⟨*proof*⟩

lemma *non-empty-cycle-root-msc-plus*:
assumes *non-empty-cycle-root* r x
shows $x^+;r = x^{T+};r$
⟨*proof*⟩

lemma *non-empty-cycle-root-tc-start-points*:
assumes *non-empty-cycle-root* r x
shows $x^+;r = x;1$
⟨*proof*⟩

lemma *non-empty-cycle-root-rtc-start-points*:

assumes *non-empty-cycle-root r x*
shows $x^\star;r = x;1$
⟨*proof*⟩

lemma *non-empty-cycle-root-converse-start-end-points*:
assumes *non-empty-cycle-root r x*
shows $x^T \leq x;1;x$
⟨*proof*⟩

lemma *non-empty-cycle-root-start-end-points-plus*:
assumes *non-empty-cycle-root r x*
shows $x;1;x \leq x^+$
⟨*proof*⟩

lemma *non-empty-cycle-root-converse-plus*:
assumes *non-empty-cycle-root r x*
shows $x^T \leq x^+$
⟨*proof*⟩

lemma *non-empty-cycle-root-plus-converse*:
assumes *non-empty-cycle-root r x*
shows $x^+ = x^{T+}$
⟨*proof*⟩

lemma *non-empty-cycle-root-converse*:
assumes *non-empty-cycle-root r x*
shows *non-empty-cycle-root r* (x^T)
⟨*proof*⟩

lemma *non-empty-cycle-root-move-root-forward*:
assumes *non-empty-cycle-root r x*
and *point q*
and $r \leq x^\star;q$
shows *non-empty-cycle-root q x*
⟨*proof*⟩

lemma *non-empty-cycle-root-move-root-forward-cycle*:
assumes *non-empty-cycle-root r x*
and *point q*
and $r \leq x^\star;q$
shows $x;q \neq 0 \wedge x^T;q \neq 0$
⟨*proof*⟩

lemma *non-empty-cycle-root-equivalences*:
assumes *non-empty-cycle-root r x*
and *point q*
shows $(r \leq x^\star;q \longleftrightarrow q \leq x^\star;r)$
and $(r \leq x^\star;q \longleftrightarrow x;q \neq 0)$
and $(r \leq x^\star;q \longleftrightarrow x^T;q \neq 0)$

and $(r \leq x^\star;q \longleftrightarrow q \leq x;1)$
and $(r \leq x^\star;q \longleftrightarrow q \leq x^T;1)$
⟨*proof*⟩

lemma *non-empty-cycle-root-chord*:
assumes *non-empty-cycle-root r x*
and *point p*
and *point q*
and $r \leq x^\star;p$
and $r \leq x^\star;q$
shows $p \leq x^\star;q$
⟨*proof*⟩

lemma *non-empty-cycle-root-var-axioms-2*:
non-empty-cycle-root r x $\longleftrightarrow x;1 \leq x^+;r \wedge$ *is-inj x* $\wedge$ *is-p-fun x* $\wedge$ *point r* $\wedge\ r \leq x;1$
⟨*proof*⟩

lemma *non-empty-cycle-root-var-axioms-3*:
non-empty-cycle-root r x $\longleftrightarrow x;1 \leq x^+;r \wedge$ *is-inj x* $\wedge$ *is-p-fun x* $\wedge$ *point r* $\wedge\ r \leq x^+;x;1$
⟨*proof*⟩

lemma *non-empty-cycle-root-subset-equals*:
assumes *non-empty-cycle-root r x*
and *non-empty-cycle-root r y*
and $x \leq y$
shows $x = y$
⟨*proof*⟩

lemma *non-empty-cycle-root-subset-equals-change-root*:
assumes *non-empty-cycle-root r x*
and *non-empty-cycle-root q y*
and $x \leq y$
shows $x = y$
⟨*proof*⟩

lemma *non-empty-cycle-root-equivalences-2*:
assumes *non-empty-cycle-root r x*
shows $(v \leq x^\star;r \longleftrightarrow v \leq x^T;1)$
and $(v \leq x^\star;r \longleftrightarrow v \leq x;1)$
⟨*proof*⟩

lemma *cycle-root-non-empty*:
assumes $x \neq 0$
shows *cycle-root r x* $\longleftrightarrow$ *non-empty-cycle-root r x*
⟨*proof*⟩

Start points and end points

lemma *start-points-path-aux*:
assumes *backward-finite-path-root* r x
and *start-points* $x \neq 0$
shows $x;r = 0$
⟨*proof*⟩

lemma *start-points-path*:
assumes *backward-finite-path-root* r x
and *start-points* $x \neq 0$
shows *backward-terminating-path-root* r x
⟨*proof*⟩

lemma *root-in-start-points-2*:
assumes *backward-finite-path-root* r x
and *start-points* $x \neq 0$
shows $r \leq$ *start-points* x
⟨*proof*⟩

lemma *root-equals-start-points-2*:
assumes *backward-finite-path-root* r x
and *start-points* $x \neq 0$
shows $r =$ *start-points* x
⟨*proof*⟩

lemma *start-points-injective*:
assumes *backward-finite-path-root* r x
shows *is-inj* (*start-points* x)
⟨*proof*⟩

lemma *backward-terminating-path-root-aux-2*:
assumes *backward-finite-path-root* r x
and *start-points* $x \neq 0 \vee x = 0$
shows $x \leq x^{T\star};-(x^T;1)$
⟨*proof*⟩

lemma *start-points-not-zero-iff*:
assumes *backward-finite-path-root* r x
shows $x;r = 0 \wedge x \neq 0 \longleftrightarrow$ *start-points* $x \neq 0$
⟨*proof*⟩

Backwards terminating and backwards finite: Part II

lemma *backward-finite-path-root-acyclic-terminating-aux*:
assumes *backward-finite-path-root* r x
and *is-acyclic* x
shows $x;r = 0$
⟨*proof*⟩

lemma *backward-finite-path-root-acyclic-terminating-iff*:
assumes *backward-finite-path-root* r x

shows *is-acyclic* $x \longleftrightarrow x;r = 0$
⟨*proof*⟩

lemma *backward-finite-path-root-acyclic-terminating*:
assumes *backward-finite-path-root* $r\ x$
and *is-acyclic* x
shows *backward-terminating-path-root* $r\ x$
⟨*proof*⟩

lemma *non-empty-cycle-root-one-strongly-connected*:
assumes *non-empty-cycle-root* $r\ x$
shows *one-strongly-connected* x
⟨*proof*⟩

lemma *backward-finite-path-root-nodes-reachable*:
assumes *backward-finite-path-root* $r\ x$
and $v \le x;1 + x^T;1$
and *is-sur* v
shows $r \le x^\star;v$
⟨*proof*⟩

lemma *terminating-path-root-end-backward-terminating*:
assumes *terminating-path-root-end* $r\ x\ e$
shows *backward-terminating-path-root* $r\ x$
⟨*proof*⟩

lemma *terminating-path-root-end-converse*:
assumes *terminating-path-root-end* $r\ x\ e$
shows *terminating-path-root-end* $e\ (x^T)\ r$
⟨*proof*⟩

lemma *terminating-path-root-end-forward-terminating*:
assumes *terminating-path-root-end* $r\ x\ e$
shows *backward-terminating-path-root* $e\ (x^T)$
⟨*proof*⟩

end

3.3 Consequences with the Tarski rule and the point axiom

context *relation-algebra-rtc-tarski-point*
begin

Rooted paths

lemma *path-root-iff*:
$(\exists\, r\ .\ \textit{path-root}\ r\ x) \longleftrightarrow \textit{path}\ x$
⟨*proof*⟩

lemma *non-empty-path-root-iff*:
$(\exists\, r\ .\ \textit{path-root}\ r\ x \wedge r \le (x + x^T);1) \longleftrightarrow \textit{path}\ x \wedge x \ne 0$

⟨*proof*⟩

(Non-empty) Cycle

lemma *non-empty-cycle-root-iff*:
$(\exists r\ .\ \textit{non-empty-cycle-root}\ r\ x) \longleftrightarrow \textit{cycle}\ x \wedge x \neq \mathit{0}$
⟨*proof*⟩

lemma *non-empty-cycle-subset-equals*:
assumes *cycle x*
and *cycle y*
and $x \leq y$
and $x \neq \mathit{0}$
shows $x = y$
⟨*proof*⟩

lemma *cycle-root-iff*:
$(\exists r\ .\ \textit{cycle-root}\ r\ x) \longleftrightarrow \textit{cycle}\ x$
⟨*proof*⟩

Backwards terminating and backwards finite

lemma *backward-terminating-path-root-iff*:
$(\exists r\ .\ \textit{backward-terminating-path-root}\ r\ x) \longleftrightarrow \textit{backward-terminating-path}\ x$
⟨*proof*⟩

lemma *non-empty-backward-terminating-path-root-iff*:
$\textit{backward-terminating-path-root}\ (\textit{start-points}\ x)\ x \longleftrightarrow \textit{backward-terminating-path}\ x \wedge x \neq \mathit{0}$
⟨*proof*⟩

lemma *non-empty-backward-terminating-path-root-iff'*:
$\textit{backward-finite-path-root}\ (\textit{start-points}\ x)\ x \longleftrightarrow \textit{backward-terminating-path}\ x \wedge x \neq \mathit{0}$
⟨*proof*⟩

lemma *backward-finite-path-root-iff*:
$(\exists r\ .\ \textit{backward-finite-path-root}\ r\ x) \longleftrightarrow \textit{backward-finite-path}\ x$
⟨*proof*⟩

lemma *non-empty-backward-finite-path-root-iff*:
$(\exists r\ .\ \textit{backward-finite-path-root}\ r\ x \wedge r \leq x;\mathit{1}) \longleftrightarrow \textit{backward-finite-path}\ x \wedge x \neq \mathit{0}$
⟨*proof*⟩

Terminating

lemma *terminating-path-root-end-aux*:
assumes *terminating-path x*
shows $\exists r\ e\ .\ \textit{terminating-path-root-end}\ r\ x\ e$
⟨*proof*⟩

lemma *terminating-path-root-end-iff*:
$(\exists\, r\ e\ .\ \mathit{terminating\text{-}path\text{-}root\text{-}end}\ r\ x\ e) \longleftrightarrow \mathit{terminating\text{-}path}\ x$
⟨*proof*⟩

lemma *non-empty-terminating-path-root-end-iff*:
$\mathit{terminating\text{-}path\text{-}root\text{-}end}\ (\mathit{start\text{-}points}\ x)\ x\ (\mathit{end\text{-}points}\ x) \longleftrightarrow \mathit{terminating\text{-}path}\ x \wedge x \neq 0$
⟨*proof*⟩

lemma *non-empty-finite-path-root-end-iff*:
$\mathit{finite\text{-}path\text{-}root\text{-}end}\ (\mathit{start\text{-}points}\ x)\ x\ (\mathit{end\text{-}points}\ x) \longleftrightarrow \mathit{terminating\text{-}path}\ x \wedge x \neq 0$
⟨*proof*⟩

end

end

4 Correctness of Path Algorithms

To show that our theory of paths integrates with verification tasks, we verify the correctness of three basic path algorithms. Algorithms at the presented level are executable and can serve prototyping purposes. Data refinement can be carried out to move from such algorithms to more efficient programs. The total-correctness proofs use a library developed in [7].

theory *Path-Algorithms*

imports *HOL−Hoare.Hoare-Logic Rooted-Paths*

begin

unbundle *no trancl-syntax*

class *choose-singleton-point-signature* =
fixes *choose-singleton* :: $'a \Rightarrow 'a$
fixes *choose-point* :: $'a \Rightarrow 'a$

class *relation-algebra-rtc-tarski-choose-point* =
relation-algebra-rtc-tarski + *choose-singleton-point-signature* +
assumes *choose-singleton-singleton*: $x \neq 0 \Longrightarrow \mathit{singleton}\ (\mathit{choose\text{-}singleton}\ x)$
assumes *choose-singleton-decreasing*: $\mathit{choose\text{-}singleton}\ x \leq x$
assumes *choose-point-point*: $\mathit{is\text{-}vector}\ x \Longrightarrow x \neq 0 \Longrightarrow \mathit{point}\ (\mathit{choose\text{-}point}\ x)$
assumes *choose-point-decreasing*: $\mathit{choose\text{-}point}\ x \leq x$
begin

no-notation
composition (**infixl** ‹;› *75*) **and**
times (**infixl** ‹∗› *70*)

notation
composition (**infixl** ‹∗› *75*)

4.1 Construction of a path

Our first example is a basic greedy algorithm that constructs a path from a vertex x to a different vertex y of a directed acyclic graph D.

abbreviation *construct-path-inv* $q\ x\ y\ D\ W \equiv$
$\quad$ *is-acyclic* $D \wedge$ *point* $x \wedge$ *point* $y \wedge$ *point* $q \wedge$
$\quad D^\star * q \leq D^{T\star} * x \wedge W \leq D \wedge$ *terminating-path* $W \wedge$
$\quad (W = 0 \longleftrightarrow q{=}y) \wedge (W \neq 0 \longleftrightarrow q =$ *start-points* $W \wedge y =$ *end-points* $W)$

abbreviation *construct-path-inv-simp* $q\ x\ y\ D\ W \equiv$
$\quad$ *is-acyclic* $D \wedge$ *point* $x \wedge$ *point* $y \wedge$ *point* $q \wedge$
$\quad D^\star * q \leq D^{T\star} * x \wedge W \leq D \wedge$ *terminating-path* $W \wedge$
$\quad q =$ *start-points* $W \wedge y =$ *end-points* W

lemma *construct-path-pre*:
  assumes *is-acyclic* D
    and *point* y
    and *point* x
    and $D^\star * y \leq D^{T\star} * x$
  shows *construct-path-inv* $y\ x\ y\ D\ 0$
⟨*proof*⟩

The following three lemmas are auxiliary lemmas for *construct-path-inv*. They are pulled out of the main proof to have more structure.

lemma *path-inv-points*:
  assumes *construct-path-inv* $q\ x\ y\ D\ W \wedge q \neq x$
  shows *point* q
    and *point* (*choose-point* $(D{*}q)$)
⟨*proof*⟩

lemma *path-inv-choose-point-decrease*:
  assumes *construct-path-inv* $q\ x\ y\ D\ W \wedge q \neq x$
  shows $W{\neq}0 \Longrightarrow$ *choose-point* $(D{*}q) \leq -((W +$ *choose-point* $(D{*}q) * q^T)^T{*}1)$
⟨*proof*⟩

lemma *end-points*:
  assumes *construct-path-inv* $q\ x\ y\ D\ W \wedge q \neq x$
  shows *choose-point* $(D{*}q) =$ *start-points* $(W +$ *choose-point* $(D{*}q) * q^T)$
    and $y =$ *end-points* $(W +$ *choose-point* $(D{*}q) * q^T)$
⟨*proof*⟩

lemma *construct-path-inv*:
  assumes *construct-path-inv* $q\ x\ y\ D\ W \wedge q \neq x$

shows *construct-path-inv* (*choose-point* ($D*q$)) x y D (W + *choose-point* ($D*q$)$*q^T$)
⟨*proof*⟩

theorem *construct-path-partial*: *VARS* p q W
{ *is-acyclic* D ∧ *point* y ∧ *point* x ∧ $D^\star * y \leq D^{T\star} * x$ }
W := *0*;
q := y;
WHILE $q \neq x$
INV { *construct-path-inv* q x y D W }
DO p := *choose-point* ($D*q$);
W := W + $p*q^T$;
q := p
OD
{ $W \leq D$ ∧ *terminating-path* W ∧ (W=*0* ⟷ x=y) ∧ (W≠*0* ⟷ x = *start-points* W ∧ y = *end-points* W) }
⟨*proof*⟩

end

For termination, we additionally need finiteness.

context *finite*
begin

lemma *decrease-set*:
assumes ∀ x::′*a* . Q x ⟶ P x
and P w
and ¬ Q w
shows *card* { x . Q x } < *card* { x . P x }
⟨*proof*⟩

end

class *relation-algebra-rtc-tarski-choose-point-finite* =
relation-algebra-rtc-tarski-choose-point +
relation-algebra-rtc-tarski-point-finite
begin

lemma *decrease-variant*:
assumes $y \leq z$
and $w \leq z$
and ¬ $w \leq y$
shows *card* { x . $x \leq y$ } < *card* { x . $x \leq z$ }
⟨*proof*⟩

lemma *construct-path-inv-termination*:
assumes *construct-path-inv* q x y D W ∧ $q \neq x$
shows *card* { z . $z \leq -(W$ + *choose-point* ($D*q$)$*q^T)$ } < *card* { z . $z \leq -W$ }

⟨*proof*⟩

theorem *construct-path-total*: *VARS p q W*
[*is-acyclic D* $\wedge$ *point y* $\wedge$ *point x* $\wedge$ $D^\star * y \leq D^{T\star} * x$]
W := *0*;
q := *y*;
WHILE $q \neq x$
INV { *construct-path-inv q x y D W* }
VAR { *card* { z . $z \leq -W$ } }
DO *p* := *choose-point* $(D * q)$;
$W := W + p * q^T$;
$q := p$
OD
[$W \leq D$ $\wedge$ *terminating-path W* $\wedge$ $(W = 0 \longleftrightarrow x = y)$ $\wedge$ $(W \neq 0 \longleftrightarrow x =$ *start-points W* $\wedge$ $y =$ *end-points W*)]
⟨*proof*⟩

end

4.2 Topological sorting

In our second example we look at topological sorting. Given a directed acyclic graph, the problem is to construct a linear order of its vertices that contains x before y for each edge (x, y) of the graph. If the input graph models dependencies between tasks, the output is a linear schedule of the tasks that respects all dependencies.

context *relation-algebra-rtc-tarski-choose-point*
begin

abbreviation *topological-sort-inv*
where *topological-sort-inv q v R W* $\equiv$
regressively-finite R $\wedge$ $R \cdot v * v^T \leq W^+$ $\wedge$ *terminating-path W* $\wedge$ $W * 1 = v \cdot -q$ $\wedge$
$(W = 0 \vee q =$ *end-points W*) $\wedge$ *point q* $\wedge$ $R * v \leq v$ $\wedge$ $q \leq v$ $\wedge$ *is-vector v*

lemma *topological-sort-pre*:
assumes *regressively-finite R*
shows *topological-sort-inv* (*choose-point* (*minimum R 1*)) (*choose-point* (*minimum R 1*)) *R 0*
⟨*proof*⟩

lemma *topological-sort-inv*:
assumes $v \neq 1$
and *topological-sort-inv q v R W*
shows *topological-sort-inv* (*choose-point* (*minimum R* $(-v)$)) ($v +$ *choose-point* (*minimum R* $(-v)$)) *R* ($W + q *$ *choose-point* (*minimum R* $(-v))^T$)

⟨proof⟩

lemma *topological-sort-post*:
assumes $\neg\ v \neq 1$
and *topological-sort-inv q v R W*
shows $R \leq W^+ \wedge$ *terminating-path* $W \wedge (W + W^T)*1 = -1'*1$
⟨proof⟩

theorem *topological-sort-partial*: *VARS p q v W*
{ *regressively-finite R* }
$W := 0$;
$q :=$ *choose-point* (*minimum R 1*);
$v := q$;
WHILE $v \neq 1$
INV { *topological-sort-inv q v R W* }
DO $p :=$ *choose-point* (*minimum R* $(-v)$);
$W := W + q*p^T$;
$q := p$;
$v := v + p$
OD
{ $R \leq W^+ \wedge$ *terminating-path* $W \wedge (W + W^T)*1 = -1'*1$ }
⟨proof⟩

end

context *relation-algebra-rtc-tarski-choose-point-finite*
begin

lemma *topological-sort-inv-termination*:
assumes $v \neq 1$
and *topological-sort-inv q v R W*
shows *card* $\{z . z \leq -(v +$ *choose-point* (*minimum R* $(-v)))\}$ < *card* $\{ z . z \leq -v \}$
⟨proof⟩

Use precondition *is-acyclic* instead of *regressively-finite*. They are equivalent for finite graphs.

theorem *topological-sort-total*: *VARS p q v W*
[*is-acyclic R*]
$W := 0$;
$q :=$ *choose-point* (*minimum R 1*);
$v := q$;
WHILE $v \neq 1$
INV { *topological-sort-inv q v R W* }
VAR { *card* $\{ z . z \leq -v \}$ }
DO $p :=$ *choose-point* (*minimum R* $(-v)$);
$W := W + q*p^T$;
$q := p$;
$v := v + p$

OD
[$R \leq W^{+} \wedge$ *terminating-path* $W \wedge (W + W^{T}) * 1 = -1' * 1$]
⟨*proof*⟩

end

4.3 Construction of a tree

Our last application is a correctness proof of an algorithm that constructs a non-empty cycle for a given directed graph. This works in two steps. The first step is to construct a directed tree from a given root along the edges of the graph.

context *relation-algebra-rtc-tarski-choose-point*
begin

abbreviation *construct-tree-pre*
where *construct-tree-pre* $x\ y\ R \equiv y \leq R^{T\star} * x \wedge$ *point* x
abbreviation *construct-tree-inv*
where *construct-tree-inv* $v\ x\ y\ D\ R \equiv$ *construct-tree-pre* $x\ y\ R \wedge$ *is-acyclic* $D \wedge$ *is-inj* $D \wedge$
$D \leq R \wedge D * x = 0 \wedge v = x + D^{T} * 1 \wedge x * v^{T} \leq D^{\star} \wedge D \leq v * v^{T} \wedge$
is-vector v
abbreviation *construct-tree-post*
where *construct-tree-post* $x\ y\ D\ R \equiv$ *is-acyclic* $D \wedge$ *is-inj* $D \wedge D \leq R \wedge D * x = 0 \wedge D^{T} * 1 \leq D^{T\star} * x \wedge$
$D^{\star} * y \leq D^{T\star} * x$

lemma *construct-tree-pre*:
assumes *construct-tree-pre* $x\ y\ R$
shows *construct-tree-inv* $x\ x\ y\ 0\ R$
⟨*proof*⟩

lemma *construct-tree-inv-aux*:
assumes $\neg\ y \leq v$
and *construct-tree-inv* $v\ x\ y\ D\ R$
shows *singleton* (*choose-singleton* $(v * -v^{T} \cdot R)$)
⟨*proof*⟩

lemma *construct-tree-inv*:
assumes $\neg\ y \leq v$
and *construct-tree-inv* $v\ x\ y\ D\ R$
shows *construct-tree-inv* (v + *choose-singleton* $(v * -v^{T} \cdot R)^{T} * 1$) $x\ y$ (D + *choose-singleton* $(v * -v^{T} \cdot R)$) R
⟨*proof*⟩

lemma *construct-tree-post*:
assumes $y \leq v$
and *construct-tree-inv* $v\ x\ y\ D\ R$

shows *construct-tree-post* $x\ y\ D\ R$
⟨*proof*⟩

theorem *construct-tree-partial*: *VARS* $e\ v\ D$
{ *construct-tree-pre* $x\ y\ R$ }
$D := 0$;
$v := x$;
WHILE $\neg\ y \leq v$
INV { *construct-tree-inv* $v\ x\ y\ D\ R$ }
DO $e :=$ *choose-singleton* $(v{*}{-}v^T \cdot R)$;
$D := D + e$;
$v := v + e^T{*}1$
OD
{ *construct-tree-post* $x\ y\ D\ R$ }
⟨*proof*⟩

end

context *relation-algebra-rtc-tarski-choose-point-finite*
begin

lemma *construct-tree-inv-termination*:
assumes $\neg\ y \leq v$
and *construct-tree-inv* $v\ x\ y\ D\ R$
shows *card* $\{\ z\ .\ z \leq -(v +$ *choose-singleton* $(v{*}{-}v^T \cdot R)^T{*}1)\ \} <$ *card* $\{\ z\ .\ z \leq -v\ \}$
⟨*proof*⟩

theorem *construct-tree-total*: *VARS* $e\ v\ D$
[*construct-tree-pre* $x\ y\ R$]
$D := 0$;
$v := x$;
WHILE $\neg\ y \leq v$
INV { *construct-tree-inv* $v\ x\ y\ D\ R$ }
VAR { *card* $\{\ z\ .\ z \leq -v\ \}$ }
DO $e :=$ *choose-singleton* $(v{*}{-}v^T \cdot R)$;
$D := D + e$;
$v := v + e^T{*}1$
OD
[*construct-tree-post* $x\ y\ D\ R$]
⟨*proof*⟩

end

4.4 Construction of a non-empty cycle

The second step is to construct a path from the root to a given vertex in the tree. Adding an edge back to the root gives the cycle.

context *relation-algebra-rtc-tarski-choose-point*

begin

abbreviation *comment*
where $\mathit{comment}\ \text{-} \equiv \mathit{SKIP}$
abbreviation *construct-cycle-inv*
where $\mathit{construct\text{-}cycle\text{-}inv}\ v\ x\ y\ D\ R \equiv \mathit{construct\text{-}tree\text{-}inv}\ v\ x\ y\ D\ R \wedge \mathit{point}\ y \wedge y{*}x^T \leq R$

lemma *construct-cycle-pre*:
assumes $\neg\ \mathit{is\text{-}acyclic}\ R$
and $y = \mathit{choose\text{-}point}\ ((R^+ \cdot \mathit{1'}){*}\mathit{1})$
and $x = \mathit{choose\text{-}point}\ (R^\star{*}y \cdot R^T{*}y)$
shows $\mathit{construct\text{-}cycle\text{-}inv}\ x\ x\ y\ \mathit{0}\ R$
$\langle \mathit{proof} \rangle$

lemma *construct-cycle-pre2*:
assumes $y \leq v$
and $\mathit{construct\text{-}cycle\text{-}inv}\ v\ x\ y\ D\ R$
shows $\mathit{construct\text{-}path\text{-}inv}\ y\ x\ y\ D\ \mathit{0} \wedge D \leq R \wedge D * x = \mathit{0} \wedge y * x^T \leq R$
$\langle \mathit{proof} \rangle$

lemma *construct-cycle-post*:
assumes $\neg\ q \neq x$
and $(\mathit{construct\text{-}path\text{-}inv}\ q\ x\ y\ D\ W \wedge D \leq R \wedge D * x = \mathit{0} \wedge y * x^T \leq R)$
shows $W + y * x^T \neq \mathit{0} \wedge W + y * x^T \leq R \wedge \mathit{cycle}\ (W + y * x^T)$
$\langle \mathit{proof} \rangle$

theorem *construct-cycle-partial*: $\mathit{VARS}\ e\ p\ q\ v\ x\ y\ C\ D\ W$
$\{\ \neg\ \mathit{is\text{-}acyclic}\ R\ \}$
$y := \mathit{choose\text{-}point}\ ((R^+ \cdot \mathit{1'}){*}\mathit{1});$
$x := \mathit{choose\text{-}point}\ (R^\star{*}y \cdot R^T{*}y);$
$D := \mathit{0};$
$v := x;$
$\mathit{WHILE}\ \neg\ y \leq v$
$\mathit{INV}\ \{\ \mathit{construct\text{-}cycle\text{-}inv}\ v\ x\ y\ D\ R\ \}$
$\mathit{DO}\ e := \mathit{choose\text{-}singleton}\ (v{*}{-}v^T \cdot R);$
$D := D + e;$
$v := v + e^T{*}\mathit{1}$
$\mathit{OD};$
$\mathit{comment}\ \{\ \mathit{is\text{-}acyclic}\ D \wedge \mathit{point}\ y \wedge \mathit{point}\ x \wedge D^\star{*}y \leq D^{T\star}{*}x\ \};$
$W := \mathit{0};$
$q := y;$
$\mathit{WHILE}\ q \neq x$
$\mathit{INV}\ \{\ \mathit{construct\text{-}path\text{-}inv}\ q\ x\ y\ D\ W \wedge D \leq R \wedge D{*}x = \mathit{0} \wedge y{*}x^T \leq R\ \}$
$\mathit{DO}\ p := \mathit{choose\text{-}point}\ (D{*}q);$
$W := W + p{*}q^T;$
$q := p$
$\mathit{OD};$
$\mathit{comment}\ \{\ W \leq D \wedge \mathit{terminating\text{-}path}\ W \wedge (W = \mathit{0} \longleftrightarrow q{=}y) \wedge (W \neq \mathit{0}$

$\longleftrightarrow$ *q = start-points W* $\wedge$ *y = end-points W*) };
  C := *W* + $y{*}x^T$
  { $C \neq 0 \wedge C \leq R \wedge$ *cycle C* }
  ⟨*proof*⟩

end

context *relation-algebra-rtc-tarski-choose-point-finite*
begin

theorem *construct-cycle-total*: *VARS e p q v x y C D W*
  [¬ *is-acyclic R*]
  y := *choose-point* $((R^+ \cdot 1') {*} 1)$;
  x := *choose-point* $(R^\star {*} y \cdot R^T {*} y)$;
  D := *0*;
  v := *x*;
  WHILE ¬ $y \leq v$
    INV { *construct-cycle-inv v x y D R* }
    VAR { *card* { $z . z \leq -v$ } }
     DO e := *choose-singleton* $(v{*}{-}v^T \cdot R)$;
        D := *D* + *e*;
        v := *v* + $e^T{*}1$
     OD;
  comment { *is-acyclic D* $\wedge$ *point y* $\wedge$ *point x* $\wedge$ $D^\star{*}y \leq D^{T\star}{*}x$ };
  W := *0*;
  q := *y*;
  WHILE $q \neq x$
    INV { *construct-path-inv q x y D W* $\wedge$ $D \leq R \wedge D{*}x = 0 \wedge y{*}x^T \leq R$ }
    VAR { *card* { $z . z \leq -W$ } }
     DO p := *choose-point* $(D{*}q)$;
        W := *W* + $p{*}q^T$;
        q := *p*
     OD;
  comment { $W \leq D \wedge$ *terminating-path W* $\wedge$ ($W = 0 \longleftrightarrow q{=}y$) $\wedge$ ($W \neq 0$
$\longleftrightarrow$ *q = start-points W* $\wedge$ *y = end-points W*)};
  C := *W* + $y{*}x^T$
 [$C \neq 0 \wedge C \leq R \wedge$ *cycle C*]
  ⟨*proof*⟩

end

end

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