Public Announcement Logic

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Abstract

This work is a formalization of public announcement logic with countably many agents. It includes proofs of soundness and completeness for variants of the axiom system PA + DIST! + NEC! [1]. The completeness proofs build on the Epistemic Logic theory.

Contents

theory *PAL* **imports** *Epistemic-Logic.Epistemic-Logic* **begin**

1 Syntax

datatype *'i pfm*
  = *FF* (‹⊥!›)
  | *Pro' id* (‹Pro!›)
  | *Dis* ‹'i pfm› ‹'i pfm› (**infixr** ‹∨!› 60)
  | *Con* ‹'i pfm› ‹'i pfm› (**infixr** ‹∧!› 65)
  | *Imp* ‹'i pfm› ‹'i pfm› (**infixr** ‹⟶!› 55)
  | *K' 'i* ‹'i pfm› (‹K!›)
  | *Ann* ‹'i pfm› ‹'i pfm› (‹[-]! -› [80, 80] 80)

abbreviation *PIff* :: ‹'i pfm ⇒ 'i pfm ⇒ 'i pfm› (**infixr** ‹⟷!› 55) **where**
  ‹p ⟷! q ≡ (p ⟶! q) ∧! (q ⟶! p)›

abbreviation *PNeg* (‹¬! -› [70] 70) **where**
  ‹¬! p ≡ p ⟶! ⊥!›

abbreviation *PL* (‹L!›) **where**
  ‹L! i p ≡ (¬! (K! i (¬! p)))›

primrec *anns* :: ‹'i pfm ⇒ 'i pfm set› **where**
  ‹anns ⊥! = {}›
| ‹anns (Pro! -) = {}›
| ‹anns (p ∨! q) = (anns p ∪ anns q)›
| ‹anns (p ∧! q) = (anns p ∪ anns q)›
| ‹anns (p ⟶! q) = (anns p ∪ anns q)›
| ‹anns (K! i p) = anns p›
| ‹anns ([r]! p) = {r} ∪ anns r ∪ anns p›

2 Semantics

fun
  psemantics :: ‹('i, 'w) kripke ⇒ 'w ⇒ 'i pfm ⇒ bool› (‹-, - ⊨! -› [50, 50, 50] 50) **and**
  restrict :: ‹('i, 'w) kripke ⇒ 'i pfm ⇒ ('i, 'w) kripke› (‹-[-!]› [50, 50] 50) **where**
  ‹M, w ⊨! ⊥! ⟷ False›
| ‹M, w ⊨! Pro! x ⟷ π M w x›
| ‹M, w ⊨! p ∨! q ⟷ M, w ⊨! p ∨ M, w ⊨! q›
| ‹M, w ⊨! p ∧! q ⟷ M, w ⊨! p ∧ M, w ⊨! q›
| ‹M, w ⊨! p ⟶! q ⟷ M, w ⊨! p ⟶ M, w ⊨! q›
| ‹M, w ⊨! K! i p ⟷ (∀ v ∈ 𝒲 M ∩ 𝒦 M i w. M, v ⊨! p)›
| ‹M, w ⊨! [r]! p ⟷ M, w ⊨! r ⟶ M[r!], w ⊨! p›
| ‹M[r!] = M ⦇𝒲 := {w. w ∈ 𝒲 M ∧ M, w ⊨! r}⦈›

abbreviation *validPStar* :: ‹(('i, 'w) kripke ⇒ bool) ⇒ 'i pfm set ⇒ 'i pfm ⇒ bool›
(‹-; - $\Vdash_!\star$ -› [50, 50, 50] 50) **where**
‹P; G $\Vdash_!\star$ $p \equiv \forall M.\ P\ M \longrightarrow (\forall w \in \mathcal{W}\ M.\ (\forall q \in G.\ M, w \models_! q) \longrightarrow M, w \models_! p)$›

primrec *static* :: ‹'i pfm ⇒ bool› **where**
‹*static* $\bot_!$ = *True*›
| ‹*static* (*Pro*$_!$ -) = *True*›
| ‹*static* ($p \vee_! q$) = (*static* p ∧ *static* q)›
| ‹*static* ($p \wedge_! q$) = (*static* p ∧ *static* q)›
| ‹*static* ($p \longrightarrow_! q$) = (*static* p ∧ *static* q)›
| ‹*static* ($K_!\ i\ p$) = *static* p›
| ‹*static* ($[r]_!\ p$) = *False*›

primrec *lower* :: ‹'i pfm ⇒ 'i fm› **where**
‹*lower* $\bot_!$ = $\bot$›
| ‹*lower* (*Pro*$_!$ x) = *Pro* x›
| ‹*lower* ($p \vee_! q$) = (*lower* p ∨ *lower* q)›
| ‹*lower* ($p \wedge_! q$) = (*lower* p ∧ *lower* q)›
| ‹*lower* ($p \longrightarrow_! q$) = (*lower* $p \longrightarrow$ *lower* q)›
| ‹*lower* ($K_!\ i\ p$) = $K\ i$ (*lower* p)›
| ‹*lower* ($[r]_!\ p$) = *undefined*›

primrec *lift* :: ‹'i fm ⇒ 'i pfm› **where**
‹*lift* $\bot$ = $\bot_!$›
| ‹*lift* (*Pro* x) = *Pro*$_!$ x›
| ‹*lift* ($p \vee q$) = (*lift* $p \vee_!$ *lift* q)›
| ‹*lift* ($p \wedge q$) = (*lift* $p \wedge_!$ *lift* q)›
| ‹*lift* ($p \longrightarrow q$) = (*lift* $p \longrightarrow_!$ *lift* q)›
| ‹*lift* ($K\ i\ p$) = $K_!\ i$ (*lift* p)›

lemma *lower-semantics*:
assumes ‹*static* p›
shows ‹($M, w \models$ *lower* p) ⟷ ($M, w \models_! p$)›
⟨*proof*⟩

lemma *lift-semantics*: ‹($M, w \models p$) ⟷ ($M, w \models_!$ *lift* p)›
⟨*proof*⟩

lemma *lower-lift*: ‹*lower* (*lift* p) = p›
⟨*proof*⟩

lemma *lift-lower*: ‹*static* p ⟹ *lift* (*lower* p) = p›
⟨*proof*⟩

3 Soundness of Reduction

primrec *reduce'* :: ‹'i pfm ⇒ 'i pfm ⇒ 'i pfm› **where**

$\langle reduce'\ r\ \bot_! = (r \longrightarrow_! \bot_!)\rangle$
$|\ \langle reduce'\ r\ (Pro_!\ x) = (r \longrightarrow_! Pro_!\ x)\rangle$
$|\ \langle reduce'\ r\ (p \vee_! q) = (reduce'\ r\ p \vee_! reduce'\ r\ q)\rangle$
$|\ \langle reduce'\ r\ (p \wedge_! q) = (reduce'\ r\ p \wedge_! reduce'\ r\ q)\rangle$
$|\ \langle reduce'\ r\ (p \longrightarrow_! q) = (reduce'\ r\ p \longrightarrow_! reduce'\ r\ q)\rangle$
$|\ \langle reduce'\ r\ (K_!\ i\ p) = (r \longrightarrow_! K_!\ i\ (reduce'\ r\ p))\rangle$
$|\ \langle reduce'\ r\ ([p]_!\ q) = undefined\rangle$

primrec *reduce* :: $\langle 'i\ pfm \Rightarrow 'i\ pfm\rangle$ **where**
$\langle reduce\ \bot_! = \bot_!\rangle$
$|\ \langle reduce\ (Pro_!\ x) = Pro_!\ x\rangle$
$|\ \langle reduce\ (p \vee_! q) = (reduce\ p \vee_! reduce\ q)\rangle$
$|\ \langle reduce\ (p \wedge_! q) = (reduce\ p \wedge_! reduce\ q)\rangle$
$|\ \langle reduce\ (p \longrightarrow_! q) = (reduce\ p \longrightarrow_! reduce\ q)\rangle$
$|\ \langle reduce\ (K_!\ i\ p) = K_!\ i\ (reduce\ p)\rangle$
$|\ \langle reduce\ ([r]_!\ p) = reduce'\ (reduce\ r)\ (reduce\ p)\rangle$

lemma *static-reduce'*: $\langle static\ p \Longrightarrow static\ r \Longrightarrow static\ (reduce'\ r\ p)\rangle$
$\langle proof\rangle$

lemma *static-reduce*: $\langle static\ (reduce\ p)\rangle$
$\langle proof\rangle$

lemma *reduce'-semantics*:
assumes $\langle static\ q\rangle$
shows $\langle (M,\ w \models_! [p]_!\ q) = (M,\ w \models_! reduce'\ p\ q)\rangle$
$\langle proof\rangle$

lemma *reduce-semantics*: $\langle M,\ w \models_! p \longleftrightarrow M,\ w \models_! reduce\ p\rangle$
$\langle proof\rangle$

4 Chains of Implications

primrec *implyP* :: $\langle 'i\ pfm\ list \Rightarrow 'i\ pfm \Rightarrow 'i\ pfm\rangle$ (**infixr** $\langle \leadsto_! \rangle$ *56*) **where**
$\langle ([] \leadsto_! q) = q\rangle$
$|\ \langle (p\ \#\ ps \leadsto_! q) = (p \longrightarrow_! ps \leadsto_! q)\rangle$

lemma *lift-implyP*: $\langle lift\ (ps \leadsto q) = (map\ lift\ ps \leadsto_! lift\ q)\rangle$
$\langle proof\rangle$

lemma *reduce-implyP*: $\langle reduce\ (ps \leadsto_! q) = (map\ reduce\ ps \leadsto_! reduce\ q)\rangle$
$\langle proof\rangle$

5 Proof System

primrec *peval* :: $\langle (id \Rightarrow bool) \Rightarrow ('i\ pfm \Rightarrow bool) \Rightarrow 'i\ pfm \Rightarrow bool\rangle$ **where**
$\langle peval\ \text{-}\ \text{-}\ \bot_! = False\rangle$
$|\ \langle peval\ g\ \text{-}\ (Pro_!\ x) = g\ x\rangle$

$|\ \langle peval\ g\ h\ (p \vee_! q) = (peval\ g\ h\ p \vee peval\ g\ h\ q)\rangle$
$|\ \langle peval\ g\ h\ (p \wedge_! q) = (peval\ g\ h\ p \wedge peval\ g\ h\ q)\rangle$
$|\ \langle peval\ g\ h\ (p \longrightarrow_! q) = (peval\ g\ h\ p \longrightarrow peval\ g\ h\ q)\rangle$
$|\ \langle peval\ \text{-}\ h\ (K_!\ i\ p) = h\ (K_!\ i\ p)\rangle$
$|\ \langle peval\ \text{-}\ h\ ([r]_!\ p) = h\ ([r]_!\ p)\rangle$

abbreviation $\langle ptautology\ p \equiv \forall\, g\ h.\ peval\ g\ h\ p\rangle$

inductive $PAK :: \langle('i\ pfm \Rightarrow bool) \Rightarrow ('i\ pfm \Rightarrow bool) \Rightarrow 'i\ pfm \Rightarrow bool\rangle$
$(\langle\text{-};\ \text{-} \vdash_!\ \text{-}\rangle\ [50,\ 50,\ 50]\ 50)$
for $A\ B :: \langle 'i\ pfm \Rightarrow bool\rangle$ **where**
$PA1\colon \langle ptautology\ p \Longrightarrow A;\ B \vdash_! p\rangle$
$|\ PA2\colon \langle A;\ B \vdash_! K_!\ i\ p \wedge_! K_!\ i\ (p \longrightarrow_! q) \longrightarrow_! K_!\ i\ q\rangle$
$|\ PAx\colon \langle A\ p \Longrightarrow A;\ B \vdash_! p\rangle$
$|\ PR1\colon \langle A;\ B \vdash_! p \Longrightarrow A;\ B \vdash_! p \longrightarrow_! q \Longrightarrow A;\ B \vdash_! q\rangle$
$|\ PR2\colon \langle A;\ B \vdash_! p \Longrightarrow A;\ B \vdash_! K_!\ i\ p\rangle$
$|\ PAnn\colon \langle A;\ B \vdash_! p \Longrightarrow B\ r \Longrightarrow A;\ B \vdash_! [r]_!\ p\rangle$
$|\ PFF\colon \langle A;\ B \vdash_! [r]_!\ \bot_! \longleftrightarrow_! (r \longrightarrow_! \bot_!)\rangle$
$|\ PPro\colon \langle A;\ B \vdash_! [r]_!\ Pro_!\ x \longleftrightarrow_! (r \longrightarrow_! Pro_!\ x)\rangle$
$|\ PDis\colon \langle A;\ B \vdash_! [r]_!\ (p \vee_! q) \longleftrightarrow_! [r]_!\ p \vee_! [r]_!\ q\rangle$
$|\ PCon\colon \langle A;\ B \vdash_! [r]_!\ (p \wedge_! q) \longleftrightarrow_! [r]_!\ p \wedge_! [r]_!\ q\rangle$
$|\ PImp\colon \langle A;\ B \vdash_! [r]_!\ (p \longrightarrow_! q) \longleftrightarrow_! ([r]_!\ p \longrightarrow_! [r]_!\ q)\rangle$
$|\ PK\colon \langle A;\ B \vdash_! [r]_!\ K_!\ i\ p \longleftrightarrow_! (r \longrightarrow_! K_!\ i\ ([r]_!\ p))\rangle$

abbreviation $PAK\text{-}assms\ (\langle\text{-};\ \text{-};\ \text{-} \vdash_!\ \text{-}\rangle\ [50,\ 50,\ 50,\ 50]\ 50)$ **where**
$\langle A;\ B;\ G \vdash_! p \equiv \exists\, qs.\ set\ qs \subseteq G \wedge (A;\ B \vdash_! qs \rightsquigarrow_! p)\rangle$

lemma $eval\text{-}peval\colon \langle eval\ h\ (g\ o\ lift)\ p = peval\ h\ g\ (lift\ p)\rangle$
$\langle proof\rangle$

lemma $tautology\text{-}ptautology\colon \langle tautology\ p \Longrightarrow ptautology\ (lift\ p)\rangle$
$\langle proof\rangle$

theorem $AK\text{-}PAK\colon \langle A\ o\ lift \vdash p \Longrightarrow A;\ B \vdash_! lift\ p\rangle$
$\langle proof\rangle$

abbreviation $validP$
$:: \langle(('i,\ 'i\ fm\ set)\ kripke \Rightarrow bool) \Rightarrow 'i\ pfm\ set \Rightarrow 'i\ pfm \Rightarrow bool\rangle$
$(\langle\text{-};\ \text{-} \models_!\ \text{-}\rangle\ [50,\ 50,\ 50]\ 50)$
where $\langle P;\ G \models_! p \equiv P;\ G \models_{!\star} p\rangle$

lemma $set\text{-}map\text{-}inv$:
assumes $\langle set\ xs \subseteq f\ \text{`}\ X\rangle$
shows $\langle \exists\, ys.\ set\ ys \subseteq X \wedge map\ f\ ys = xs\rangle$
$\langle proof\rangle$

lemma $strong\text{-}static\text{-}completeness'$:
assumes $\langle static\ p\rangle$ **and** $\langle \forall\, q \in G.\ static\ q\rangle$ **and** $\langle P;\ G \models_! p\rangle$
and $\langle P;\ lower\ \text{`}\ G \models_\star lower\ p \Longrightarrow A\ o\ lift;\ lower\ \text{`}\ G \vdash lower\ p\rangle$

shows $\langle A;\ B;\ G \vdash_! p\rangle$
$\langle proof \rangle$

theorem *strong-static-completeness*:
assumes $\langle static\ p\rangle$ **and** $\langle\forall\, q \in G.\ static\ q\rangle$ **and** $\langle P;\ G \models_! p\rangle$
and $\langle\bigwedge G\ p.\ P;\ G \models p \Longrightarrow A\ o\ lift;\ G \vdash p\rangle$
shows $\langle A;\ B;\ G \vdash_! p\rangle$
$\langle proof \rangle$

corollary *static-completeness'*:
assumes $\langle static\ p\rangle$ **and** $\langle P;\ \{\} \models_{!\star} p\rangle$
and $\langle P;\ \{\} \models lower\ p \Longrightarrow A\ o\ lift \vdash lower\ p\rangle$
shows $\langle A;\ B \vdash_! p\rangle$
$\langle proof \rangle$

corollary *static-completeness*:
assumes $\langle static\ p\rangle$ **and** $\langle P;\ \{\} \models_{!\star} p\rangle$ **and** $\langle\bigwedge p.\ P;\ \{\} \models p \Longrightarrow A\ o\ lift \vdash p\rangle$
shows $\langle A;\ B \vdash_! p\rangle$
$\langle proof \rangle$

corollary
assumes $\langle static\ p\rangle$ $\langle(\lambda\text{-}.\ True);\ \{\} \models_! p\rangle$
shows $\langle A;\ B \vdash_! p\rangle$
$\langle proof \rangle$

6 Soundness

lemma *peval-semantics*:
$\langle peval\ (val\ w)\ (\lambda q.\ (\!|\mathcal{W} = W,\ \mathcal{K} = r,\ \pi = val|\!),\ w \models_! q)\ p = ((\!|\mathcal{W} = W,\ \mathcal{K} = r,\ \pi = val|\!),\ w \models_! p)\rangle$
$\langle proof \rangle$

lemma *ptautology*:
assumes $\langle ptautology\ p\rangle$
shows $\langle M,\ w \models_! p\rangle$
$\langle proof \rangle$

theorem $soundness_P$:
assumes
$\langle\bigwedge M\ p\ w.\ A\ p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M,\ w \models_! p\rangle$
$\langle\bigwedge M\ r.\ P\ M \Longrightarrow B\ r \Longrightarrow P\ (M[r!])\rangle$
shows $\langle A;\ B \vdash_! p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M,\ w \models_! p\rangle$
$\langle proof \rangle$

corollary $\langle(\lambda\text{-}.\ False);\ B \vdash_! p \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M,\ w \models_! p\rangle$
$\langle proof \rangle$

7 Strong Soundness

lemma *ptautology-imply-superset*:
assumes ‹$set\ ps \subseteq set\ qs$›
shows ‹$ptautology\ (ps \rightsquigarrow_! r \longrightarrow_! qs \rightsquigarrow_! r)$›
⟨*proof*⟩

lemma *PK-imply-weaken*:
assumes ‹$A;\ B \vdash_! ps \rightsquigarrow_! q$› ‹$set\ ps \subseteq set\ ps'$›
shows ‹$A;\ B \vdash_! ps' \rightsquigarrow_! q$›
⟨*proof*⟩

lemma *implyP-append*: ‹$(ps\ @\ ps' \rightsquigarrow_! q) = (ps \rightsquigarrow_! ps' \rightsquigarrow_! q)$›
⟨*proof*⟩

lemma *PK-ImpI*:
assumes ‹$A;\ B \vdash_! p\ \#\ G \rightsquigarrow_! q$›
shows ‹$A;\ B \vdash_! G \rightsquigarrow_! (p \longrightarrow_! q)$›
⟨*proof*⟩

corollary *soundness-imply$_P$*:
assumes
‹$\bigwedge M\ p\ w.\ A\ p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M,\ w \models_! p$›
‹$\bigwedge M\ r.\ P\ M \Longrightarrow B\ r \Longrightarrow P\ (M[r!])$›
shows ‹$A;\ B \vdash_! qs \rightsquigarrow_! p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow \forall\, q \in set\ qs.\ M,\ w \models_! q \Longrightarrow M,\ w \models_! p$›
⟨*proof*⟩

theorem *strong-soundness$_P$*:
assumes
‹$\bigwedge M\ w\ p.\ A\ p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M,\ w \models_! p$›
‹$\bigwedge M\ r.\ P\ M \Longrightarrow B\ r \Longrightarrow P\ (M[r!])$›
shows ‹$A;\ B;\ G \vdash_! p \Longrightarrow P;\ G \models_{!\star} p$›
⟨*proof*⟩

8 Completeness

lemma *ConE*:
assumes ‹$A;\ B \vdash_! p \wedge_! q$›
shows ‹$A;\ B \vdash_! p$› ‹$A;\ B \vdash_! q$›
⟨*proof*⟩

lemma *Iff-Dis*:
assumes ‹$A;\ B \vdash_! p \longleftrightarrow_! p'$› ‹$A;\ B \vdash_! q \longleftrightarrow_! q'$›
shows ‹$A;\ B \vdash_! ((p \vee_! q) \longleftrightarrow_! (p' \vee_! q'))$›
⟨*proof*⟩

lemma *Iff-Con*:
assumes ‹$A;\ B \vdash_! p \longleftrightarrow_! p'$› ‹$A;\ B \vdash_! q \longleftrightarrow_! q'$›

shows ‹A; $B \vdash_! (p \wedge_! q) \longleftrightarrow_! (p' \wedge_! q')$›
⟨*proof*⟩

lemma *Iff-Imp*:
assumes ‹A; $B \vdash_! p \longleftrightarrow_! p'$› ‹$A$; $B \vdash_! q \longleftrightarrow_! q'$›
shows ‹A; $B \vdash_! ((p \longrightarrow_! q) \longleftrightarrow_! (p' \longrightarrow_! q'))$›
⟨*proof*⟩

lemma *Iff-sym*: ‹(A; $B \vdash_! p \longleftrightarrow_! q) = (A$; $B \vdash_! q \longleftrightarrow_! p)$›
⟨*proof*⟩

lemma *Iff-Iff*:
assumes ‹A; $B \vdash_! p \longleftrightarrow_! p'$› ‹$A$; $B \vdash_! p \longleftrightarrow_! q$›
shows ‹A; $B \vdash_! p' \longleftrightarrow_! q$›
⟨*proof*⟩

lemma *K'-A2'*: ‹A; $B \vdash_! K_!\ i\ (p \longrightarrow_! q) \longrightarrow_! K_!\ i\ p \longrightarrow_! K_!\ i\ q$›
⟨*proof*⟩

lemma *K'-map*:
assumes ‹A; $B \vdash_! p \longrightarrow_! q$›
shows ‹A; $B \vdash_! K_!\ i\ p \longrightarrow_! K_!\ i\ q$›
⟨*proof*⟩

lemma *ConI*:
assumes ‹A; $B \vdash_! p$› ‹A; $B \vdash_! q$›
shows ‹A; $B \vdash_! p \wedge_! q$›
⟨*proof*⟩

lemma *Iff-wk*:
assumes ‹A; $B \vdash_! p \longleftrightarrow_! q$›
shows ‹A; $B \vdash_! (r \longrightarrow_! p) \longleftrightarrow_! (r \longrightarrow_! q)$›
⟨*proof*⟩

lemma *Iff-reduce'*:
assumes ‹*static p*›
shows ‹A; $B \vdash_! [r]_!\ p \longleftrightarrow_!$ *reduce' r p*›
⟨*proof*⟩

lemma *Iff-Ann1*:
assumes r: ‹A; $B \vdash_! r \longleftrightarrow_! r'$› **and** ‹*static p*›
shows ‹A; $B \vdash_! [r]_!\ p \longleftrightarrow_! [r']_!\ p$›
⟨*proof*⟩

lemma *Iff-Ann2*:
assumes ‹A; $B \vdash_! p \longleftrightarrow_! p'$› **and** ‹$B\ r$›
shows ‹A; $B \vdash_! [r]_!\ p \longleftrightarrow_! [r]_!\ p'$›
⟨*proof*⟩

lemma *Iff-reduce*:
  assumes ‹$\forall r \in anns\ p.\ B\ r$›
  shows ‹$A;\ B \vdash_! p \longleftrightarrow_! reduce\ p$›
  ⟨*proof*⟩

lemma *anns-implyP* [*simp*]:
  ‹$anns\ (ps \leadsto_! q) = \ anns\ q \cup (\bigcup p \in set\ ps.\ anns\ p)$›
  ⟨*proof*⟩

lemma *strong-completeness*$_P$′:
  assumes ‹$P;\ G \models_! p$›
    and ‹$\forall r \in anns\ p.\ B\ r$› ‹$\forall q \in G.\ \forall r \in anns\ q.\ B\ r$›
    and ‹$P;\ lower$ ‘ $reduce$ ‘ $G \models\star lower\ (reduce\ p) \Longrightarrow$
      $A\ o\ lift;\ lower$ ‘ $reduce$ ‘ $G \vdash lower\ (reduce\ p)$›
  shows ‹$A;\ B;\ G \vdash_! p$›
⟨*proof*⟩

theorem *strong-completeness*$_P$:
  assumes ‹$P;\ G \models_! p$›
    and ‹$\forall r \in anns\ p.\ B\ r$› ‹$\forall q \in G.\ \forall r \in anns\ q.\ B\ r$›
    and ‹$\bigwedge G\ p.\ P;\ G \models\star p \Longrightarrow A\ o\ lift;\ G \vdash p$›
  shows ‹$A;\ B;\ G \vdash_! p$›
  ⟨*proof*⟩

theorem *main*$_P$:
  assumes ‹$\bigwedge M\ w\ p.\ A\ p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M,\ w \models_! p$›
    and ‹$\bigwedge M\ r.\ P\ M \Longrightarrow B\ r \Longrightarrow P\ (M[r!])$›
    and ‹$\forall r \in anns\ p.\ B\ r$› ‹$\forall q \in G.\ \forall r \in anns\ q.\ B\ r$›
    and ‹$\bigwedge G\ p.\ P;\ G \models\star p \Longrightarrow A\ o\ lift;\ G \vdash p$›
  shows ‹$P;\ G \models_! p \longleftrightarrow A;\ B;\ G \vdash_! p$›
  ⟨*proof*⟩

corollary *strong-completeness*$_{PB}$:
  assumes ‹$P;\ G \models_! p$›
    and ‹$\bigwedge G\ p.\ P;\ G \models\star p \Longrightarrow A\ o\ lift;\ G \vdash p$›
  shows ‹$A;\ (\lambda\text{-}.\ True);\ G \vdash_! p$›
  ⟨*proof*⟩

corollary *completeness*$_P$′:
  assumes ‹$P;\ \{\} \models_! p$›
    and ‹$\forall r \in anns\ p.\ B\ r$›
    and ‹$\bigwedge p.\ P;\ \{\} \models lower\ p \Longrightarrow A\ o\ lift \vdash lower\ p$›
  shows ‹$A;\ B \vdash_! p$›
  ⟨*proof*⟩

corollary *completeness*$_P$:
  assumes ‹$P;\ \{\} \models_! p$›
    and ‹$\forall r \in anns\ p.\ B\ r$›
    and ‹$\bigwedge p.\ P;\ \{\} \models p \Longrightarrow A\ o\ lift \vdash p$›

shows ‹A; $B \vdash_! p$›
⟨*proof*⟩

corollary *completeness*$_{PA}$:
assumes ‹(λ-. *True*); {} $\models_! p$›
shows ‹A; (λ-. *True*) $\vdash_! p$›
⟨*proof*⟩

9 System PAL + K

abbreviation *SystemPK* (‹- $\vdash_{!K}$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{!K} p \equiv$ (λ-. *False*); (λ-. *True*); $G \vdash_! p$›

lemma *strong-soundness*$_{PK}$: ‹$G \vdash_{!K} p \Longrightarrow$ (λ-. *True*); $G \models_{!\star} p$›
⟨*proof*⟩

abbreviation *validPK* (‹- $\models_{!K}$ -› [*50*, *50*] *50*) **where**
‹$G \models_{!K} p \equiv$ (λ-. *True*); $G \models_! p$›

lemma *strong-completeness*$_{PK}$:
assumes ‹$G \models_{!K} p$›
shows ‹$G \vdash_{!K} p$›
⟨*proof*⟩

theorem *main*$_{PK}$: ‹$G \models_{!K} p \longleftrightarrow G \vdash_{!K} p$›
⟨*proof*⟩

corollary ‹$G \models_{!K} p \Longrightarrow$ (λ-. *True*); $G \models_{!\star} p$›
⟨*proof*⟩

10 System PAL + T

Also known as System PAL + M

inductive *AxPT* :: ‹*'i pfm* ⇒ *bool*› **where**
‹*AxPT* ($K_!$ i $p \longrightarrow_! p$)›

abbreviation *SystemPT* (‹- $\vdash_{!T}$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{!T} p \equiv$ *AxPT*; (λ-. *True*); $G \vdash_! p$›

lemma *soundness-AxPT*: ‹*AxPT* $p \Longrightarrow$ *reflexive* $M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models_!$ p›
⟨*proof*⟩

lemma *reflexive-restrict*: ‹*reflexive* $M \Longrightarrow$ *reflexive* ($M[r!]$)›
⟨*proof*⟩

lemma *strong-soundness*$_{PT}$: ‹$G \vdash_{!T} p \Longrightarrow$ *reflexive*; $G \models_{!\star} p$›

⟨*proof*⟩

lemma *AxT-AxPT*: ‹*AxT* = *AxPT o lift*›
⟨*proof*⟩

abbreviation *validPT* (‹- $\models_{!T}$ -› [*50*, *50*] *50*) **where**
‹$G \models_{!T} p \equiv$ *reflexive*; $G \models_{!} p$›

lemma *strong-completeness*$_{PT}$:
assumes ‹$G \models_{!T} p$›
shows ‹$G \vdash_{!T} p$›
⟨*proof*⟩

theorem *main*$_{PT}$: ‹$G \models_{!T} p \longleftrightarrow G \vdash_{!T} p$›
⟨*proof*⟩

corollary ‹$G \models_{!T} p \Longrightarrow$ *reflexive*; $G \models_{!\star} p$›
⟨*proof*⟩

11 System PAL + KB

inductive *AxPB* :: ‹*'i pfm* ⇒ *bool*› **where**
‹*AxPB* ($p \longrightarrow_{!} K_{!}\ i\ (L_{!}\ i\ p)$)›

abbreviation *SystemPKB* (‹- $\vdash_{!KB}$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{!KB} p \equiv$ *AxPB*; (λ-. *True*); $G \vdash_{!} p$›

lemma *soundness-AxPB*: ‹*AxPB* $p \Longrightarrow$ *symmetric* $M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models_{!} p$›
⟨*proof*⟩

lemma *symmetric-restrict*: ‹*symmetric* $M \Longrightarrow$ *symmetric* ($M[r!]$)›
⟨*proof*⟩

lemma *strong-soundness*$_{PKB}$: ‹$G \vdash_{!KB} p \Longrightarrow$ *symmetric*; $G \models_{!\star} p$›
⟨*proof*⟩

lemma *AxB-AxPB*: ‹*AxB* = *AxPB o lift*›
⟨*proof*⟩

abbreviation *validPKB* (‹- $\models_{!KB}$ -› [*50*, *50*] *50*) **where**
‹$G \models_{!KB} p \equiv$ *symmetric*; $G \models_{!} p$›

lemma *strong-completeness*$_{PKB}$:
assumes ‹$G \models_{!KB} p$›
shows ‹$G \vdash_{!KB} p$›
⟨*proof*⟩

theorem *main*$_{PKB}$: ‹$G \models_{!KB} p \longleftrightarrow G \vdash_{!KB} p$›

$\langle proof \rangle$

corollary ‹$G \models_{!KB}\ p \Longrightarrow$ *symmetric*; $G \models_{!\star}\ p$›
$\langle proof \rangle$

12 System PAL + K4

inductive *AxP4* :: ‹*'i pfm* ⇒ *bool*› **where**
‹$AxP4\ (K_!\ i\ p \longrightarrow_!\ K_!\ i\ (K_!\ i\ p))$›

abbreviation *SystemPK4* (‹- $\vdash_{!K4}$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{!K4}\ p \equiv AxP4; (\lambda\text{-}.\ True); G \vdash_!\ p$›

lemma *pos-introspection*:
assumes ‹*transitive M*› ‹$w \in \mathcal{W}\ M$›
shows ‹$M, w \models_!\ (K_!\ i\ p \longrightarrow_!\ K_!\ i\ (K_!\ i\ p))$›
$\langle proof \rangle$

lemma *soundness-AxP4*: ‹$AxP4\ p \Longrightarrow transitive\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models_!$ p›
$\langle proof \rangle$

lemma *transitive-restrict*: ‹$transitive\ M \Longrightarrow transitive\ (M[r!])$›
$\langle proof \rangle$

lemma $strong\text{-}soundness_{PK4}$: ‹$G \vdash_{!K4}\ p \Longrightarrow$ *transitive*; $G \models_{!\star}\ p$›
$\langle proof \rangle$

lemma *Ax4-AxP4*: ‹$Ax4 = AxP4\ o\ lift$›
$\langle proof \rangle$

abbreviation *validPK4* (‹- $\models_{!K4}$ -› [*50*, *50*] *50*) **where**
‹$G \models_{!K4}\ p \equiv$ *transitive*; $G \models_!\ p$›

lemma $strong\text{-}completeness_{PK4}$:
assumes ‹$G \models_{!K4}\ p$›
shows ‹$G \vdash_{!K4}\ p$›
$\langle proof \rangle$

theorem $main_{PK4}$: ‹$G \models_{!K4}\ p \longleftrightarrow G \vdash_{!K4}\ p$›
$\langle proof \rangle$

corollary ‹$G \models_{!K4}\ p \Longrightarrow$ *transitive*; $G \models_{!\star}\ p$›
$\langle proof \rangle$

13 System PAL + K5

inductive *AxP5* :: ‹*'i pfm* ⇒ *bool*› **where**

‹$AxP5$ $(L_!\ i\ p \longrightarrow_! K_!\ i\ (L_!\ i\ p))$›

abbreviation *SystemPK5* (‹- $\vdash_{!K5}$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{!K5} p \equiv AxP5$; (λ-. *True*); $G \vdash_! p$›

lemma *soundness-AxP5*: ‹$AxP5\ p \Longrightarrow Euclidean\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models_! p$›
⟨*proof*⟩

lemma *Euclidean-restrict*: ‹$Euclidean\ M \Longrightarrow Euclidean\ (M[r!])$›
⟨*proof*⟩

lemma *strong-soundness*$_{PK5}$: ‹$G \vdash_{!K5} p \Longrightarrow Euclidean; G \models_{!\star} p$›
⟨*proof*⟩

lemma *Ax5-AxP5*: ‹$Ax5 = AxP5\ o\ lift$›
⟨*proof*⟩

abbreviation *validPK5* (‹- $\models_{!K5}$ -› [*50*, *50*] *50*) **where**
‹$G \models_{!K5} p \equiv Euclidean; G \models_! p$›

lemma *strong-completeness*$_{PK5}$:
assumes ‹$G \models_{!K5} p$›
shows ‹$G \vdash_{!K5} p$›
⟨*proof*⟩

theorem *main*$_{PK5}$: ‹$G \models_{!K5} p \longleftrightarrow G \vdash_{!K5} p$›
⟨*proof*⟩

corollary ‹$G \models_{!K5} p \Longrightarrow Euclidean; G \models_{!\star} p$›
⟨*proof*⟩

14 System PAL + S4

abbreviation *SystemPS4* (‹- $\vdash_{!S4}$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{!S4} p \equiv AxPT \oplus AxP4$; (λ-. *True*); $G \vdash_! p$›

lemma *soundness-AxPT4*: ‹$(AxPT \oplus AxP4)\ p \Longrightarrow refltrans\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models_! p$›
⟨*proof*⟩

lemma *refltrans-restrict*: ‹$refltrans\ M \Longrightarrow refltrans\ (M[r!])$›
⟨*proof*⟩

lemma *strong-soundness*$_{PS4}$: ‹$G \vdash_{!S4} p \Longrightarrow refltrans; G \models_{!\star} p$›
⟨*proof*⟩

lemma *AxT4-AxPT4*: ‹$(AxT \oplus Ax4) = (AxPT \oplus AxP4)\ o\ lift$›
⟨*proof*⟩

abbreviation *validPS4* (‹- $\Vdash_{!S4}$ -› [*50*, *50*] *50*) **where**
‹$G \Vdash_{!S4} p \equiv$ *refltrans*; $G \Vdash_{!} p$›

theorem *strong-completeness*$_{PS4}$:
assumes ‹$G \Vdash_{!S4} p$›
shows ‹$G \vdash_{!S4} p$›
⟨*proof*⟩

theorem *main*$_{PS4}$: ‹$G \Vdash_{!S4} p \longleftrightarrow G \vdash_{!S4} p$›
⟨*proof*⟩

corollary ‹$G \Vdash_{!S4} p \Longrightarrow$ *refltrans*; $G \Vdash_{!\star} p$›
⟨*proof*⟩

15 System PAL + S5

abbreviation *SystemPS5* (‹- $\vdash_{!S5}$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{!S5} p \equiv AxPT \oplus AxPB \oplus AxP4$; ($\lambda$-. *True*); $G \vdash_{!} p$›

abbreviation *AxPTB4* :: ‹′*i pfm* ⇒ *bool*› **where**
‹$AxPTB4 \equiv AxPT \oplus AxPB \oplus AxP4$›

lemma *soundness-AxPTB4*: ‹$AxPTB4\ p \Longrightarrow$ *equivalence* $M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow$ $M, w \models_{!} p$›
⟨*proof*⟩

lemma *equivalence-restrict*: ‹*equivalence* $M \Longrightarrow$ *equivalence* ($M[r!]$)›
⟨*proof*⟩

lemma *strong-soundness*$_{PS5}$: ‹$G \vdash_{!S5} p \Longrightarrow$ *equivalence*; $G \Vdash_{!\star} p$›
⟨*proof*⟩

lemma *AxTB4-AxPTB4*: ‹$AxTB4 = AxPTB4$ *o lift*›
⟨*proof*⟩

abbreviation *validPS5* (‹- $\Vdash_{!S5}$ -› [*50*, *50*] *50*) **where**
‹$G \Vdash_{!S5} p \equiv$ *equivalence*; $G \Vdash_{!} p$›

theorem *strong-completeness*$_{PS5}$:
assumes ‹$G \Vdash_{!S5} p$›
shows ‹$G \vdash_{!S5} p$›
⟨*proof*⟩

theorem *main*$_{PS5}$: ‹$G \Vdash_{!S5} p \longleftrightarrow G \vdash_{!S5} p$›
⟨*proof*⟩

corollary ‹$G \Vdash_{!S5} p \Longrightarrow$ *equivalence*; $G \Vdash_{!\star} p$›
⟨*proof*⟩

16 System PAL + S5'

abbreviation *SystemPS5'* (‹- $\vdash_{!S5}$'' -› [*50*, *50*] *50*) **where**
‹$G \vdash_{!S5}{}' \; p \equiv AxPT \oplus AxP5$; (λ-. *True*); $G \vdash_! \; p$›

abbreviation *AxPT5* :: ‹'*i pfm* ⇒ *bool*› **where**
‹$AxPT5 \equiv AxPT \oplus AxP5$›

lemma *soundness-AxPT5*: ‹$AxPT5 \; p \Longrightarrow equivalence \; M \Longrightarrow w \in \mathcal{W} \; M \Longrightarrow M, w \models_! \; p$›
⟨*proof*⟩

lemma *strong-soundness*$_{PS5}$'': ‹$G \vdash_{!S5}{}' \; p \Longrightarrow equivalence$; $G \models_{!\star} \; p$›
⟨*proof*⟩

lemma *AxT5-AxPT5*: ‹$AxT5 = AxPT5 \; o \; lift$›
⟨*proof*⟩

theorem *strong-completeness*$_{PS5}$'':
assumes ‹$G \models_{!S5} \; p$›
shows ‹$G \vdash_{!S5}{}' \; p$›
⟨*proof*⟩

theorem *main*$_{PS5}$'': ‹$G \models_{!S5} \; p \longleftrightarrow G \vdash_{!S5}{}' \; p$›
⟨*proof*⟩

end

References

[1] Y. Wang and Q. Cao. On axiomatizations of public announcement logic. *Synthese*, 190(Supplement-1):103–134, 2013.