

No Faster-Than-Light Observers (GenRel)

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Abstract

We have previously verified, in the first order theory **SpecRel** of Special Relativity, that inertial observers cannot travel faster than light [1, 2]. We now prove the corresponding result for **GenRel**, the first-order theory of General Relativity. Specifically, we prove that whenever an observer m encounters another observer k (so that m and k are both present at some spacetime location x), k will necessarily be observed by m to be traveling at less than light speed.

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1 Sorts

GenRel is a 2-sorted first-order logic. This theory introduces the two sorts and proves a number of basic arithmetical results. The two sorts are Bodies (things that can move) and Quantities (used to specify coordinates, masses, etc).

```
theory Sorts
  imports Main
begin
```

1.1 Bodies

There are two types of Body: photons and observers. We do not assume a priori that these sorts are disjoint.

```
record Body =
  Ph :: bool
  Ob :: bool
```

1.2 Quantities

The quantities are assumed to form a linearly ordered field. We may sometimes need to assume that the field is also Euclidean, i.e. that square roots exist, but this is not a general requirement so it will be added later using a separate axiom class, AxEField.

```
class Quantities = linordered-field
begin
```

```
abbreviation inRangeOO :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  bool ( $\lhd$  - < - <  $\rhd$ )
  where ( $a < b < c$ )  $\equiv$  ( $a < b$ )  $\wedge$  ( $b < c$ )
```

```
abbreviation inRangeOC :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  bool ( $\lhd$  - < -  $\leq$   $\rhd$ )
  where ( $a < b \leq c$ )  $\equiv$  ( $a < b$ )  $\wedge$  ( $b \leq c$ )
```

```
abbreviation inRangeCO :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  bool ( $\lhd$   $\leq$  - <  $\rhd$ )
  where ( $a \leq b < c$ )  $\equiv$  ( $a \leq b$ )  $\wedge$  ( $b < c$ )
```

```
abbreviation inRangeCC :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  bool ( $\lhd$   $\leq$  -  $\leq$   $\rhd$ )
```

where $(a \leq b \leq c) \equiv (a \leq b) \wedge (b \leq c)$

lemma *lemLEPlus*: $a \leq b + c \longrightarrow c \geq a - b$
 $\langle proof \rangle$

lemma *lemMultPosLT1*:
assumes $(a > 0) \wedge (b \geq 0) \wedge (b < 1)$
shows $(a * b) < a$
 $\langle proof \rangle$

lemma *lemAbsRange*: $e > 0 \longrightarrow ((a-e) < b < (a+e)) \longleftrightarrow (abs$
 $(b-a) < e)$
 $\langle proof \rangle$

lemma *lemAbsNeg*: $abs\ x = abs\ (-x)$
 $\langle proof \rangle$

lemma *lemAbsNegNeg*: $abs\ (-a-b) = abs\ (a+b)$
 $\langle proof \rangle$

lemma *lemGENZGT*: $(x \geq 0) \wedge (x \neq 0) \longrightarrow x > 0$
 $\langle proof \rangle$

lemma *lemLENZLT*: $(x \leq 0) \wedge (x \neq 0) \longrightarrow x < 0$
 $\langle proof \rangle$

lemma *lemSumOfNonNegAndPos*: $x \geq 0 \wedge y > 0 \longrightarrow x+y > 0$
 $\langle proof \rangle$

lemma *lemSumOfTwoHalves*: $x = x/2 + x/2$
 $\langle proof \rangle$

lemma *lemDiffDiffAdd*: $(b-a)+(c-b) = (c-a)$
 $\langle proof \rangle$

lemma *lemSumDiffCancelMiddle*: $(a - b) + (b - c) = (a - c)$
 $\langle proof \rangle$

lemma *lemDiffSumCancelMiddle*: $(a - b) + (b + c) = (a + c)$
 $\langle proof \rangle$

lemma *lemMultPosLT*: $((0 < a) \wedge (b < c)) \longrightarrow (a*b < a*c)$
 $\langle proof \rangle$

lemma *lemMultPosLE*: $((0 < a) \wedge (b \leq c)) \longrightarrow (a*b \leq a*c)$
 $\langle proof \rangle$

lemma *lemNonNegLT*: $((0 \leq a) \wedge (b < c)) \longrightarrow (a*b \leq a*c)$
 $\langle proof \rangle$

lemma *lemMultNonNegLE*: $((0 \leq a) \wedge (b \leq c)) \longrightarrow (a*b \leq a*c)$
 $\langle proof \rangle$

abbreviation *sqr* :: $'a \Rightarrow 'a$
where *sqr* $x \equiv x*x$

abbreviation *hasRoot* :: $'a \Rightarrow bool$
where *hasRoot* $x \equiv \exists r . x = \text{sqr } r$

abbreviation *isNonNegRoot* :: $'a \Rightarrow 'a \Rightarrow bool$
where *isNonNegRoot* $x \ r \equiv (r \geq 0) \wedge (x = \text{sqr } r)$

abbreviation *hasUniqueRoot* :: $'a \Rightarrow bool$
where *hasUniqueRoot* $x \equiv \exists! r . \text{isNonNegRoot } x \ r$

abbreviation *sqrt* :: $'a \Rightarrow 'a$
where *sqrt* $x \equiv \text{THE } r . \text{isNonNegRoot } x \ r$

lemma *lemAbsIsRootOfSquare*: $\text{isNonNegRoot } (\text{sqr } x) (\text{abs } x)$
 $\langle proof \rangle$

lemma *lemSqrt*:
assumes *hasRoot* x
shows *hasUniqueRoot* x
 $\langle proof \rangle$

lemma *lemSqrMonoStrict*: **assumes** $(0 \leq u) \wedge (u < v)$
shows $(\text{sqr } u) < (\text{sqr } v)$
 $\langle proof \rangle$

lemma *lemSqrMono*: $(0 \leq u) \wedge (u \leq v) \longrightarrow (\text{sqr } u) \leq (\text{sqr } v)$
 $\langle proof \rangle$

lemma *lemSqrOrderedStrict*: $(v > 0) \wedge (\text{sqr } u < \text{sqr } v) \longrightarrow (u < v)$
 $\langle proof \rangle$

lemma *lemSqrOrdered*: $(v \geq 0) \wedge (\text{sqr } u \leq \text{sqr } v) \longrightarrow (u \leq v)$
 $\langle proof \rangle$

lemma *lemSquaredNegative*: $\text{sqr } x = \text{sqr } (-x)$
 $\langle \text{proof} \rangle$

lemma *lemSqrDiffSymmetrical*: $\text{sqr } (x - y) = \text{sqr } (y - x)$
 $\langle \text{proof} \rangle$

lemma *lemSquaresPositive*: $x \neq 0 \longrightarrow \text{sqr } x > 0$
 $\langle \text{proof} \rangle$

lemma *lemZeroRoot*: $(\text{sqr } x = 0) \longleftrightarrow (x = 0)$
 $\langle \text{proof} \rangle$

lemma *lemSqrMult*: $\text{sqr } (a * b) = (\text{sqr } a) * (\text{sqr } b)$
 $\langle \text{proof} \rangle$

lemma *lemEqualSquares*: $\text{sqr } u = \text{sqr } v \longrightarrow \text{abs } u = \text{abs } v$
 $\langle \text{proof} \rangle$

lemma *lemSqrtOfSquare*:
assumes $b = \text{sqr } a$
shows $\text{sqr } b = \text{abs } a$
 $\langle \text{proof} \rangle$

lemma *lemSquareOfSqrt*:
assumes $\text{hasRoot } b$
and $a = \text{sqr } b$
shows $\text{sqr } a = b$
 $\langle \text{proof} \rangle$

lemma *lemSqrt1*: $\text{sqr } 1 = 1$
 $\langle \text{proof} \rangle$

lemma *lemSqrt0*: $\text{sqr } 0 = 0$
 $\langle \text{proof} \rangle$

lemma *lemSqrSum*: $\text{sqr } (x + y) = (x*x) + (2*x*y) + (y*y)$
 $\langle \text{proof} \rangle$

lemma *lemQuadraticGEZero*:
assumes $\forall x. a*(\text{sqr } x) + b*x + c \geq 0$
and $a > 0$
shows $(\text{sqr } b) \leq 4*a*c$

$\langle proof \rangle$

lemma *lemSquareExistsAbove*:
 shows $\exists x > 0 . (sqr\ x) > y$
 $\langle proof \rangle$

lemma *lemSmallSquares*:
 assumes $x > 0$
 shows $\exists y > 0 . (sqr\ y < x)$
 $\langle proof \rangle$

lemma *lemSqrLT1*:
 assumes $0 < x < 1$
 shows $0 < (sqr\ x) < x$
 $\langle proof \rangle$

lemma *lemReducedBound*:
 assumes $x > 0$
 shows $\exists y > 0 . (y < x) \wedge (sqr\ y < y) \wedge (y < 1)$
 $\langle proof \rangle$

end

end

2 Points

This theory defines (1+3)-dimensional spacetime points. The first coordinate is the time coordinate, and the remaining three coordinates give the spatial component.

theory *Points*
 imports *Sorts*
begin

record *'a Point* =
 tval :: *'a*
 xval :: *'a*
 yval :: *'a*
 zval :: *'a*

```

record 'a Space =
  svalx :: 'a
  svaly :: 'a
  svalz :: 'a

```

```

abbreviation tComponent :: 'a Point  $\Rightarrow$  'a where
  tComponent p  $\equiv$  tval p

```

```

abbreviation sComponent :: 'a Point  $\Rightarrow$  'a Space where
  sComponent p  $\equiv$  ( $\mid$  svalx = xval p, svaly = yval p, svalz = zval p  $\mid$ )

```

```

abbreviation mkPoint :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  'a Point where
  mkPoint t x y z  $\equiv$  ( $\mid$  tval = t, xval = x, yval = y, zval = z  $\mid$ )

```

```

abbreviation stPoint :: 'a  $\Rightarrow$  'a Space  $\Rightarrow$  'a Point where
  stPoint t s  $\equiv$  mkPoint t (svalx s) (svaly s) (svalz s)

```

```

abbreviation mkSpace :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  'a Space where
  mkSpace x y z  $\equiv$  ( $\mid$  svalx = x, svaly = y, svalz = z  $\mid$ )

```

Points have coordinates in the field of quantities, and can be thought of as the end-points of vectors pinned to the origin. We can translate and scale them, define accumulation points, etc.

```

class Points = Quantities
begin

```

```

abbreviation moveBy :: 'a Point  $\Rightarrow$  'a Point  $\Rightarrow$  'a Point ( $\langle$ -  $\oplus$  - $\rangle$ )
where
  (p  $\oplus$  q)  $\equiv$  ( $\mid$  tval = tval p + tval q,
    xval = xval p + xval q,
    yval = yval p + yval q,
    zval = zval p + zval q  $\mid$ )

```

```

abbreviation movebackBy :: 'a Point  $\Rightarrow$  'a Point  $\Rightarrow$  'a Point ( $\langle$ -  $\ominus$  - $\rangle$ ) where
  (p  $\ominus$  q)  $\equiv$  ( $\mid$  tval = tval p - tval q,
    xval = xval p - xval q,
    yval = yval p - yval q,
    zval = zval p - zval q  $\mid$ )

```

```

abbreviation sMoveBy :: 'a Space  $\Rightarrow$  'a Space  $\Rightarrow$  'a Space ( $\langle$ -  $\oplus$  s - $\rangle$ )
where
  (p  $\oplus$  s q)  $\equiv$  ( $\mid$  svalx = svalx p + svalx q,
    svaly = svaly p + svaly q,
    svalz = svalz p + svalz q  $\mid$ )

```

$$svalz = svalz\ p + svalz\ q\)$$

abbreviation $sMovebackBy :: 'a\ Space \Rightarrow 'a\ Space \Rightarrow 'a\ Space\ (\hookleftarrow - \ominus s \rightarrow)$ **where**
 $(p \ominus s\ q) \equiv (\mid\ svalx = svalx\ p - svalx\ q,$
 $\quad\quad\quad svaly = svaly\ p - svaly\ q,$
 $\quad\quad\quad svalz = svalz\ p - svalz\ q\ \mid)$

abbreviation $scaleBy :: 'a \Rightarrow 'a\ Point \Rightarrow 'a\ Point\ (\hookleftarrow - \otimes \rightarrow)$ **where**
 $scaleBy\ a\ p \equiv (\mid\ tval = a * tval\ p,\ xval = a * xval\ p,$
 $\quad\quad\quad yval = a * yval\ p,\ zval = a * zval\ p\ \mid)$

abbreviation $sScaleBy :: 'a \Rightarrow 'a\ Space \Rightarrow 'a\ Space\ (\hookleftarrow - \otimes s \rightarrow)$ **where**
 $sScaleBy\ a\ p \equiv (\mid\ svalx = a * svalx\ p,$
 $\quad\quad\quad svaly = a * svaly\ p,$
 $\quad\quad\quad svalz = a * svalz\ p\ \mid)$

abbreviation $sOrigin :: 'a\ Space$ **where**
 $sOrigin \equiv (\mid\ svalx = 0,\ svaly = 0,\ svalz = 0\ \mid)$

abbreviation $origin :: 'a\ Point$ **where**
 $origin \equiv (\mid\ tval = 0,\ xval = 0,\ yval = 0,\ zval = 0\ \mid)$

abbreviation $tUnit :: 'a\ Point$ **where**
 $tUnit \equiv (\mid\ tval = 1,\ xval = 0,\ yval = 0,\ zval = 0\ \mid)$

abbreviation $xUnit :: 'a\ Point$ **where**
 $xUnit \equiv (\mid\ tval = 0,\ xval = 1,\ yval = 0,\ zval = 0\ \mid)$

abbreviation $yUnit :: 'a\ Point$ **where**
 $yUnit \equiv (\mid\ tval = 0,\ xval = 0,\ yval = 1,\ zval = 0\ \mid)$

abbreviation $zUnit :: 'a\ Point$ **where**
 $zUnit \equiv (\mid\ tval = 0,\ xval = 0,\ yval = 0,\ zval = 1\ \mid)$

abbreviation $timeAxis :: 'a\ Point\ set$ **where**
 $timeAxis \equiv \{ p . xval\ p = 0 \wedge yval\ p = 0 \wedge zval\ p = 0 \}$

abbreviation $onTimeAxis :: 'a\ Point \Rightarrow bool$
where $onTimeAxis\ p \equiv (p \in timeAxis)$

2.1 Squared norms and separation functions

This theory defines squared norms and separations. We do not yet define unsquared norms because we are not assuming here that quantities necessarily have square roots.

abbreviation $norm2 :: 'a \text{ Point} \Rightarrow 'a$ **where**
 $norm2\ p \equiv sqr\ (tval\ p) + sqr\ (xval\ p) + sqr\ (yval\ p) + sqr\ (zval\ p)$

abbreviation $sep2 :: 'a \text{ Point} \Rightarrow 'a \text{ Point} \Rightarrow 'a$ **where**
 $sep2\ p\ q \equiv norm2\ (p \ominus q)$

abbreviation $sNorm2 :: 'a \text{ Space} \Rightarrow 'a$ **where**
 $sNorm2\ s \equiv sqr\ (svalx\ s)$
 $\quad +\ sqr\ (svaly\ s)$
 $\quad +\ sqr\ (svalz\ s)$

abbreviation $sSep2 :: 'a \text{ Point} \Rightarrow 'a \text{ Point} \Rightarrow 'a$ **where**
 $sSep2\ p\ q \equiv sqr\ (xval\ p - xval\ q)$
 $\quad +\ sqr\ (yval\ p - yval\ q)$
 $\quad +\ sqr\ (zval\ p - zval\ q)$

abbreviation $mNorm2 :: 'a \text{ Point} \Rightarrow 'a\ (\langle \| - \| m \rangle)$
where $\| p \| m \equiv sqr\ (tval\ p) - sNorm2\ (sComponent\ p)$

2.2 Topological concepts

We will need to define topological concepts like continuity and affine approximation later, so here we define open balls and accumulation points.

abbreviation $inBall :: 'a \text{ Point} \Rightarrow 'a \Rightarrow 'a \text{ Point} \Rightarrow bool$
 $(\langle - \text{ within - of } - \rangle)$
where $inBall\ q\ \varepsilon\ p \equiv sep2\ q\ p < sqr\ \varepsilon$

abbreviation $ball :: 'a \text{ Point} \Rightarrow 'a \Rightarrow 'a \text{ Point set}$
where $ball\ q\ \varepsilon \equiv \{ p . inBall\ q\ \varepsilon\ p \}$

abbreviation $accPoint :: 'a \text{ Point} \Rightarrow 'a \text{ Point set} \Rightarrow bool$
where $accPoint\ p\ s \equiv \forall\ \varepsilon > 0. \exists\ q \in s. (p \neq q) \wedge (inBall\ q\ \varepsilon\ p)$

2.3 Lines

A line is specified by giving a point on the line, and a point (thought of as a vector) giving its direction. For these purposes it doesn't matter whether the direction is "positive" or "negative".

abbreviation $line :: 'a \text{ Point} \Rightarrow 'a \text{ Point} \Rightarrow 'a \text{ Point set}$
where $line\ base\ drtn \equiv \{ p . \exists\ \alpha . p = (base \oplus (\alpha \otimes drtn)) \}$

abbreviation $lineJoining :: 'a \text{ Point} \Rightarrow 'a \text{ Point} \Rightarrow 'a \text{ Point set}$
where $lineJoining \ p \ q \equiv line \ p \ (q \ominus p)$

abbreviation $isLine :: 'a \text{ Point set} \Rightarrow bool$
where $isLine \ l \equiv \exists \ b \ d . (l = line \ b \ d)$

abbreviation $sameLine :: 'a \text{ Point set} \Rightarrow 'a \text{ Point set} \Rightarrow bool$
where $sameLine \ l1 \ l2 \equiv ((isLine \ l1) \vee (isLine \ l2)) \wedge (l1 = l2)$

abbreviation $onLine :: 'a \text{ Point} \Rightarrow 'a \text{ Point set} \Rightarrow bool$
where $onLine \ p \ l \equiv (isLine \ l) \wedge (p \in l)$

2.4 Directions

Given any two distinct points on a line, the vector joining them can be used to specify the line's direction. The direction of a line is therefore a *set* of points/vectors. By `lemDrtn` these are all parallel

fun $drtn :: 'a \text{ Point set} \Rightarrow 'a \text{ Point set}$
where $drtn \ l = \{ \ d . \exists \ p \ q . (p \neq q) \wedge (onLine \ p \ l) \wedge (onLine \ q \ l) \wedge (d = (q \ominus p)) \}$

abbreviation $parallelLines :: 'a \text{ Point set} \Rightarrow 'a \text{ Point set} \Rightarrow bool$
where $parallelLines \ l1 \ l2 \equiv (drtn \ l1) \cap (drtn \ l2) \neq \{\}$

abbreviation $parallel :: 'a \text{ Point} \Rightarrow 'a \text{ Point} \Rightarrow bool \ (\prec - \parallel - \succ)$
where $parallel \ p \ q \equiv (\exists \ \alpha \neq 0 . p = (\alpha \otimes q))$

The "slope" of a line can be either finite or infinite. We will often need to consider these two cases separately.

abbreviation $slopeFinite :: 'a \text{ Point} \Rightarrow 'a \text{ Point} \Rightarrow bool$
where $slopeFinite \ p \ q \equiv (tval \ p \neq tval \ q)$

abbreviation $slopeInfinite :: 'a \text{ Point} \Rightarrow 'a \text{ Point} \Rightarrow bool$
where $slopeInfinite \ p \ q \equiv (tval \ p = tval \ q)$

abbreviation $lineSlopeFinite :: 'a \text{ Point set} \Rightarrow bool$
where $lineSlopeFinite \ l \equiv (\exists \ x \ y . (onLine \ x \ l) \wedge (onLine \ y \ l) \wedge (x \neq y) \wedge (slopeFinite \ x \ y))$

2.5 Slopes and slopers

We specify the slope of a line by giving the spatial component ("sloper") of the point on the line at time 1. This is defined if and only if the slope is finite. If the slope is infinite (the

line is "horizontal") we return the spatial origin. This avoids using "option" but means we need to consider carefully whether a sloper with value sOrigin indicates a truly zero slope or an infinite one.

```
fun sloper :: 'a Point  $\Rightarrow$  'a Point  $\Rightarrow$  'a Point
  where sloper p q = (if (slopeFinite p q) then ((1 / (tval p - tval
q))  $\otimes$  (p  $\ominus$  q))
                    else origin)
```

```
fun velocityJoining :: 'a Point  $\Rightarrow$  'a Point  $\Rightarrow$  'a Space
  where velocityJoining p q = sComponent (sloper p q)
```

```
fun lineVelocity :: 'a Point set  $\Rightarrow$  'a Space set
  where lineVelocity l = { v .  $\exists$  d  $\in$  drtn l . v = velocityJoining origin
d }
```

```
lemma lemNorm2Decomposition:
  shows norm2 u = sqr (tval u) + sNorm2 (sComponent u)
   $\langle$ proof $\rangle$ 
```

```
lemma lemPointDecomposition:
  shows p = (((tval p)  $\otimes$  tUnit)  $\oplus$  (((xval p)  $\otimes$  xUnit)
     $\oplus$  (((yval p)  $\otimes$  yUnit)  $\oplus$  ((zval p)  $\otimes$  zUnit))))
   $\langle$ proof $\rangle$ 
```

```
lemma lemScaleLeftSumDistrib: ((a + b)  $\otimes$  p) = ((a  $\otimes$  p)  $\oplus$  (b  $\otimes$  p))
   $\langle$ proof $\rangle$ 
```

```
lemma lemScaleLeftDiffDistrib: ((a - b)  $\otimes$  p) = ((a  $\otimes$  p)  $\ominus$  (b  $\otimes$  p))
   $\langle$ proof $\rangle$ 
```

```
lemma lemScaleAssoc: ( $\alpha$   $\otimes$  ( $\beta$   $\otimes$  p)) = (( $\alpha$  *  $\beta$ )  $\otimes$  p)
   $\langle$ proof $\rangle$ 
```

lemma *lemScaleCommute*: $(\alpha \otimes (\beta \otimes p)) = (\beta \otimes (\alpha \otimes p))$
 $\langle proof \rangle$

lemma *lemScaleDistribSum*: $(\alpha \otimes (p \oplus q)) = ((\alpha \otimes p) \oplus (\alpha \otimes q))$
 $\langle proof \rangle$

lemma *lemScaleDistribDiff*: $(\alpha \otimes (p \ominus q)) = ((\alpha \otimes p) \ominus (\alpha \otimes q))$
 $\langle proof \rangle$

lemma *lemScaleOrigin*: $(\alpha \otimes origin) = origin$
 $\langle proof \rangle$

lemma *lemMNorm2OfScaled*: $mNorm2 (scaleBy \alpha p) = (sqr \alpha) * mNorm2 p$
 $\langle proof \rangle$

lemma *lemSNorm2OfScaled*: $sNorm2 (sScaleBy \alpha p) = (sqr \alpha) * sNorm2 p$
 $\langle proof \rangle$

lemma *lemNorm2OfScaled*: $norm2 (\alpha \otimes p) = (sqr \alpha) * norm2 p$
 $\langle proof \rangle$

lemma *lemScaleSep2*: $(sqr a) * (sep2 p q) = sep2 (a \otimes p) (a \otimes q)$
 $\langle proof \rangle$

lemma *lemSScaleAssoc*: $(\alpha \otimes s (\beta \otimes s p)) = ((\alpha * \beta) \otimes s p)$
 $\langle proof \rangle$

lemma *lemScaleBall*:
assumes x within e of y
and $a \neq 0$
shows $(a \otimes x)$ within $(a * e)$ of $(a \otimes y)$
 $\langle proof \rangle$

lemma *lemScaleBallAndBoundary*:
assumes $sep2 x y \leq sqr e$
and $a \neq 0$
shows $sep2 (a \otimes x) (a \otimes y) \leq sqr (a * e)$
 $\langle proof \rangle$

lemma *lemTimeAxisIsLine: isLine timeAxis*
 $\langle proof \rangle$

lemma *lemSameLine:*
assumes $p \in \text{line } b \ d$
shows $\text{sameLine } (\text{line } b \ d) (\text{line } p \ d)$
 $\langle proof \rangle$

lemma *lemSSep2Symmetry: sSep2 p q = sSep2 q p*
 $\langle proof \rangle$

lemma *lemSep2Symmetry: sep2 p q = sep2 q p*
 $\langle proof \rangle$

lemma *lemSpatialNullImpliesSpatialOrigin:*
assumes $s\text{Norm2 } s = 0$
shows $s = s\text{Origin}$
 $\langle proof \rangle$

lemma *lemNorm2NonNeg: norm2 p ≥ 0*
 $\langle proof \rangle$

lemma *lemNullImpliesOrigin:*
assumes $\text{norm2 } p = 0$
shows $p = \text{origin}$
 $\langle proof \rangle$

lemma *lemNotOriginImpliesPosNorm2:*
assumes $p \neq \text{origin}$
shows $\text{norm2 } p > 0$
 $\langle proof \rangle$

lemma *lemNotEqualImpliesSep2Pos:*
assumes $y \neq x$
shows $\text{sep2 } y \ x > 0$
 $\langle proof \rangle$

lemma *lemBallContainsCentre:*

assumes $\varepsilon > 0$

shows x within ε of x

$\langle proof \rangle$

lemma *lemPointLimit:*

assumes $\forall \varepsilon > 0 . (v \text{ within } \varepsilon \text{ of } u)$

shows $v = u$

$\langle proof \rangle$

lemma *lemBallPopulated:*

assumes $e > 0$

shows $\exists y . (y \text{ within } e \text{ of } x) \wedge (y \neq x)$

$\langle proof \rangle$

lemma *lemBallInBall:*

assumes p within x of q

and $0 < x \leq y$

shows p within y of q

$\langle proof \rangle$

lemma *lemSmallPoints:*

assumes $e > 0$

shows $\exists a > 0 . \text{norm2 } (a \otimes p) < \text{sqr } e$

$\langle proof \rangle$

lemma *lemLineJoiningContainsEndPoints:*

assumes $l = \text{lineJoining } x \ p$

shows $\text{onLine } x \ l \wedge \text{onLine } p \ l$

$\langle proof \rangle$

lemma *lemLineAndPoints:*

assumes $p \neq q$

shows $(\text{onLine } p \ l \wedge \text{onLine } q \ l) \longleftrightarrow (l = \text{lineJoining } p \ q)$

$\langle proof \rangle$

lemma *lemLineDefinedByPair:*

assumes $x \neq p$

and $(\text{onLine } p \ l1) \wedge (\text{onLine } x \ l1)$

```

and      (onLine p l2)  $\wedge$  (onLine x l2)
shows l1 = l2
 $\langle$ proof $\rangle$ 

```

```

lemma lemDrtn:
  assumes { d1, d2 }  $\subseteq$  drtn l
  shows  $\exists \alpha \neq 0 . d2 = (\alpha \otimes d1)$ 
 $\langle$ proof $\rangle$ 

```

```

lemma lemLineDeterminedByPointAndDrtn:
  assumes (x  $\neq$  p)  $\wedge$  (p  $\in$  l1)  $\wedge$  (onLine x l1)  $\wedge$  (onLine x l2)
and      drtn l1 = drtn l2
shows    l1 = l2
 $\langle$ proof $\rangle$ 

```

```

end

```

```

end

```

3 WorldView

This theory defines worldview transformations. These form the ultimate foundation for all of GenRel's axioms.

```

theory WorldView
  imports Points
begin

```

```

class WorldView = Points +
fixes

```

```

  W :: Body  $\Rightarrow$  Body  $\Rightarrow$  'a Point  $\Rightarrow$  bool ( $\leftarrow$  sees - at  $\rightarrow$ )
begin

```

```

abbreviation ev :: Body  $\Rightarrow$  'a Point  $\Rightarrow$  Body set
  where ev h x  $\equiv$  { b . h sees b at x }

```

```

fun wvt :: Body  $\Rightarrow$  Body  $\Rightarrow$  'a Point  $\Rightarrow$  'a Point set
  where wvt m k p = { q. ( $\exists$  b . (m sees b at p))  $\wedge$  (ev m p = ev k q) }

```

```

abbreviation wvtFunc :: Body  $\Rightarrow$  Body  $\Rightarrow$  ('a Point  $\Rightarrow$  'a Point  $\Rightarrow$  bool)
  where wvtFunc m k  $\equiv$  ( $\lambda$  p q . q  $\in$  wvt m k p)

```

```

abbreviation wvtLine :: Body  $\Rightarrow$  Body  $\Rightarrow$  'a Point set  $\Rightarrow$  'a Point
set  $\Rightarrow$  bool
  where wvtLine m k l l'  $\equiv$   $\exists$  p q p' q' . (
    (wvtFunc m k p p')  $\wedge$  (wvtFunc m k q q')  $\wedge$ 
    (l = lineJoining p q)  $\wedge$  (l' = lineJoining
p' q'))

end

end

```

4 Functions

This theory characterises the various types of function (injective, bijective, etc).

```

theory Functions
  imports Points
begin

```

We do not assume a priori that all of the functions we define are well-defined or total. We therefore need to allow for functions which are only partially defined, and also for "functions" which might be multi-valued. For example, we cannot say in advance whether one observer might see another's worldline as a bifurcating structure rather than a basic single-valued trajectory.

To achieve this we'll often think of functions as relations and write " $f\ x\ y = \text{true}$ " instead of " $f\ x = y$ ". Similarly, a spacetime set S will be sometimes be expressed as its characteristic function.

```

class Functions = Points
begin

```

```

abbreviation bounded :: ('a Point  $\Rightarrow$  'a Point)  $\Rightarrow$  bool
  where bounded f  $\equiv$   $\exists$  bnd > 0 . ( $\forall$  p . (norm2 (f p)  $\leq$  bnd *
(norm2 p)))

```

```

abbreviation composeRel ::
  ('a Point  $\Rightarrow$  'a Point  $\Rightarrow$  bool)
 $\Rightarrow$  ('a Point  $\Rightarrow$  'a Point  $\Rightarrow$  bool)
 $\Rightarrow$  ('a Point  $\Rightarrow$  'a Point  $\Rightarrow$  bool)
  where (composeRel g f) p r  $\equiv$  ( $\exists$  q . ((f p q)  $\wedge$  (g q r)))

```

abbreviation *injective* :: ('a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool) $\Rightarrow$ bool
where *injective* $f \equiv \forall x1\ x2\ y1\ y2.$
 $(f\ x1\ y1 \wedge f\ x2\ y2) \wedge (x1 \neq x2) \longrightarrow (y1 \neq y2)$

abbreviation *definedAt* :: ('a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool) $\Rightarrow$ 'a Point
 $\Rightarrow$ bool
where *definedAt* $f\ x \equiv \exists y . f\ x\ y$

abbreviation *domain* :: ('a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool) $\Rightarrow$ 'a Point
set
where *domain* $f \equiv \{ x . \text{definedAt } f\ x \}$

abbreviation *total* :: ('a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool) $\Rightarrow$ bool
where *total* $f \equiv \forall x . (\text{definedAt } f\ x)$

abbreviation *surjective* :: ('a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool) $\Rightarrow$ bool
where *surjective* $f \equiv \forall y . \exists x . f\ x\ y$

abbreviation *bijjective* :: ('a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool) $\Rightarrow$ bool
where *bijjective* $f \equiv (\text{injective } f) \wedge (\text{surjective } f)$

abbreviation *invertible* :: ('a Point $\Rightarrow$ 'a Point) $\Rightarrow$ bool
where *invertible* $f \equiv \forall q . (\exists p . (f\ p = q) \wedge (\forall x. f\ x = q \longrightarrow x = p))$

fun *applyToSet* :: ('a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool) $\Rightarrow$ 'a Point *set* $\Rightarrow$ 'a
Point *set*
where *applyToSet* $f\ s = \{ q . \exists p \in s . f\ p\ q \}$

abbreviation *singleValued* :: ('a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool) $\Rightarrow$ 'a Point
 $\Rightarrow$ bool
where *singleValued* $f\ x \equiv \forall y\ z . ((f\ x\ y) \wedge (f\ x\ z)) \longrightarrow (y = z)$

abbreviation *isFunction* :: ('a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool) $\Rightarrow$ bool
where *isFunction* $f \equiv \forall x . \text{singleValued } f\ x$

abbreviation *isTotalFunction* :: ('a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool) $\Rightarrow$ bool
where *isTotalFunction* $f \equiv (\text{total } f) \wedge (\text{isFunction } f)$

fun *toFunc* :: ('a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool) $\Rightarrow$ 'a Point $\Rightarrow$ 'a Point
where *toFunc* $f\ x = (\text{SOME } y . f\ x\ y)$

fun *asFunc* :: ('a Point $\Rightarrow$ 'a Point) $\Rightarrow$ ('a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool)
where *asFunc* $f\ x\ y = (y = f\ x)$

4.1 Differentiable approximation

Here we define differentiable approximation. This will be used later when we define what it means for a worldview transformation to be "approximated" by an affine transformation.

abbreviation $\text{diffApprox} :: ('a \text{ Point} \Rightarrow 'a \text{ Point} \Rightarrow \text{bool}) \Rightarrow$
 $('a \text{ Point} \Rightarrow 'a \text{ Point} \Rightarrow \text{bool}) \Rightarrow$
 $'a \text{ Point} \Rightarrow \text{bool}$
where $\text{diffApprox } g \ f \ x \equiv (\text{definedAt } f \ x) \wedge$
 $(\forall \ \varepsilon > 0 . (\exists \ \delta > 0 . (\forall \ y .$
 $((y \text{ within } \delta \text{ of } x)$
 $\longrightarrow$
 $((\text{definedAt } f \ y) \wedge (\forall \ u \ v . (f \ y \ u \wedge g \ y \ v) \longrightarrow$
 $(\text{sep2 } v \ u) \leq (\text{sqr } \varepsilon) * \text{sep2 } y \ x)))$
 $))$

abbreviation $\text{cts} :: ('a \text{ Point} \Rightarrow 'a \text{ Point} \Rightarrow \text{bool}) \Rightarrow 'a \text{ Point} \Rightarrow \text{bool}$
where $\text{cts } f \ x \equiv \forall y . (f \ x \ y) \longrightarrow (\forall \varepsilon > 0. \exists \delta > 0.$
 $(\text{applyToSet } f \ (\text{ball } x \ \delta)) \subseteq \text{ball } y \ \varepsilon)$

fun $\text{invFunc} :: ('a \text{ Point} \Rightarrow 'a \text{ Point} \Rightarrow \text{bool}) \Rightarrow ('a \text{ Point} \Rightarrow 'a \text{ Point}$
 $\Rightarrow \text{bool})$
where $(\text{invFunc } f) \ p \ q = f \ q \ p$

lemma $\text{lemBijInv} : \text{bijective } (\text{asFunc } f) \longleftrightarrow \text{invertible } f$
 $\langle \text{proof} \rangle$

4.2 lemApproxEqualAtBase

The following lemma shows (as one would expect) that when one function differentially approximates another at a point, they take equal values at that point.

lemma $\text{lemApproxEqualAtBase} :$
assumes $\text{diffApprox } g \ f \ x$
shows $(f \ x \ y \wedge g \ x \ z) \longrightarrow (y = z)$
 $\langle \text{proof} \rangle$

lemma $\text{lemCtsOfCtsIsCts} :$
assumes $\text{cts } f \ x$
and $\forall y . (f \ x \ y) \longrightarrow (\text{cts } g \ y)$
shows $\text{cts } (\text{composeRel } g \ f) \ x$

$\langle proof \rangle$

lemma *lemInjOfInjIsInj*:
 assumes *injective f*
 and *injective g*
 shows *injective (composeRel g f)*
 $\langle proof \rangle$

lemma *lemInverseComposition*:
 assumes *h = composeRel g f*
 shows (*invFunc h*) = *composeRel (invFunc f) (invFunc g)*
 $\langle proof \rangle$

lemma *lemToFuncAsFunc*:
 assumes *isFunction f*
 and *total f*
 shows *asFunc (toFunc f) = f*
 $\langle proof \rangle$

lemma *lemAsFuncToFunc*: *toFunc (asFunc f) = f*
 $\langle proof \rangle$

end

end

5 WorldLine

This theory defines worldlines.

theory *WorldLine*
 imports *WorldView Functions*
begin

class *WorldLine* = *WorldView* + *Functions*
begin

abbreviation *wline* :: *Body* $\Rightarrow$ *Body* $\Rightarrow$ '*a Point set*
 where *wline m k* $\equiv \{ p . m \text{ sees } k \text{ at } p \}$

lemma *lemWorldLineUnderWVT*:
shows $\text{applyToSet } (\text{wvtFunc } m \ k) \ (\text{wline } m \ b) \subseteq \text{wline } k \ b$
 $\langle \text{proof} \rangle$

lemma *lemFiniteLineVelocityUnique*:
assumes $(u \in \text{lineVelocity } l) \wedge (v \in \text{lineVelocity } l)$
and $\text{lineSlopeFinite } l$
shows $u = v$
 $\langle \text{proof} \rangle$

end

end

6 Translations

This theory describes translation maps.

theory *Translations*
imports *Functions*
begin

class *Translations* = *Functions*
begin

abbreviation $\text{mkTranslation} :: 'a \text{ Point} \Rightarrow ('a \text{ Point} \Rightarrow 'a \text{ Point})$
where $(\text{mkTranslation } t) \equiv (\lambda p . (p \oplus t))$

abbreviation $\text{translation} :: ('a \text{ Point} \Rightarrow 'a \text{ Point}) \Rightarrow \text{bool}$
where $\text{translation } T \equiv \exists q . \forall p . ((T \ p) = (p \oplus q))$

lemma *lemMkTrans*: $\forall t . \text{translation } (\text{mkTranslation } t)$
 $\langle \text{proof} \rangle$

lemma *lemInverseTranslation*:
assumes $(T = \text{mkTranslation } t) \wedge (T' = \text{mkTranslation } (\text{origin } \ominus t))$
shows $(T' \circ T = \text{id}) \wedge (T \circ T' = \text{id})$
 $\langle \text{proof} \rangle$

lemma *lemTranslationSum*:
assumes *translation T*
shows $T (u \oplus v) = ((T u) \oplus v)$
 $\langle proof \rangle$

lemma *lemIdIsTranslation*: *translation id*
 $\langle proof \rangle$

lemma *lemTranslationCancel*:
assumes *translation T*
shows $((T p) \ominus (T q)) = (p \ominus q)$
 $\langle proof \rangle$

lemma *lemTranslationSwap*:
assumes *translation T*
shows $(p \oplus (T q)) = ((T p) \oplus q)$
 $\langle proof \rangle$

lemma *lemTranslationPreservesSep2*:
assumes *translation T*
shows $sep2\ p\ q = sep2\ (T\ p)\ (T\ q)$
 $\langle proof \rangle$

lemma *lemTranslationInjective*:
assumes *translation T*
shows *injective* (*asFunc T*)
 $\langle proof \rangle$

lemma *lemTranslationSurjective*:
assumes *translation T*
shows *surjective* (*asFunc T*)
 $\langle proof \rangle$

lemma *lemTranslationTotalFunction*:
assumes *translation T*
shows *isTotalFunction* (*asFunc T*)
 $\langle proof \rangle$

lemma *lemTranslationOfLine*:
assumes *translation T*
shows $(\text{applyToSet } (\text{asFunc } T) (\text{line } B \ D)) = \text{line } (T \ B) \ D$
 $\langle \text{proof} \rangle$

lemma *lemOnLineTranslation*:
assumes $(\text{translation } T) \wedge (\text{onLine } p \ l)$
shows $\text{onLine } (T \ p) (\text{applyToSet } (\text{asFunc } T) \ l)$
 $\langle \text{proof} \rangle$

lemma *lemLineJoiningTranslation*:
assumes *translation T*
shows $\text{applyToSet } (\text{asFunc } T) (\text{lineJoining } p \ q) = \text{lineJoining } (T \ p) \ (T \ q)$
 $\langle \text{proof} \rangle$

lemma *lemBallTranslation*:
assumes *translation T*
and $x \text{ within } e \text{ of } y$
shows $(T \ x) \text{ within } e \text{ of } (T \ y)$
 $\langle \text{proof} \rangle$

lemma *lemBallTranslationWithBoundary*:
assumes *translation T*
and $\text{sep2 } x \ y \leq \text{sqr } e$
shows $\text{sep2 } (T \ x) \ (T \ y) \leq \text{sqr } e$
 $\langle \text{proof} \rangle$

lemma *lemTranslationIsCts*:
assumes *translation T*
shows $\text{cts } (\text{asFunc } T) \ x$
 $\langle \text{proof} \rangle$

lemma *lemAccPointTranslation*:
assumes *translation T*
and $\text{accPoint } x \ s$

```

shows    accPoint (T x) (applyToSet (asFunc T) s)
<proof>

```

```

lemma lemInverseOfTransIsTrans:
  assumes translation T
and      T' = invFunc (asFunc T)
shows    translation (toFunc T')
<proof>

```

```

lemma lemInverseTrans:
assumes translation T
shows   $\exists T' . (\text{translation } T') \wedge (\forall p\ q . T\ p = q \longleftrightarrow T'\ q = p)$ 
<proof>

```

```

end

```

```

end

```

7 AXIOM: AxSelfMinus

This theory declares the axiom AxSelfMinus.

```

theory AxSelfMinus
  imports WorldView
begin

```

AxSelfMinus: The worldline of an observer is a subset of the time axis in their own worldview.

```

class axSelfMinus = WorldView
begin
  abbreviation axSelfMinus :: Body  $\Rightarrow$  'a Point  $\Rightarrow$  bool
    where axSelfMinus m p  $\equiv$  (m sees m at p)  $\longrightarrow$  onTimeAxis p
end

```

```

class AxSelfMinus = axSelfMinus +
  assumes AxSelfMinus :  $\forall m\ p . \text{axSelfMinus } m\ p$ 
begin
end

end

```

8 TangentLines

This theory defines tangent lines and establishes their key properties.

```
theory TangentLines
imports Translations AxSelfMinus
begin
```

At each point along the worldline of a body, we can ask what its instantaneous direction of motion is. Unfortunately we do not know a priori that the "worldline" actually has tangents. Dealing with tangent lines is one of the more complicated aspects of the main proof.

```
class TangentLines = Translations + AxSelfMinus
begin
```

```
abbreviation tangentLine :: 'a Point set  $\Rightarrow$  'a Point set  $\Rightarrow$  'a Point
 $\Rightarrow$  bool
```

```
where tangentLine l s x  $\equiv$ 
  (x  $\in$  s)  $\wedge$  (onLine x l)  $\wedge$  (accPoint x s)
 $\wedge$ 
  ( $\exists$  p . ( (onLine p l)  $\wedge$  (p  $\neq$  x)  $\wedge$ 
    ( $\forall$   $\varepsilon > 0$  .  $\exists$   $\delta > 0$  .  $\forall$  y  $\in$  s . (
      (y within  $\delta$  of x)  $\wedge$  (y  $\neq$  x) )
     $\longrightarrow$ 
    ( $\exists$  r . ((onLine r (lineJoining x y))  $\wedge$  (r within  $\varepsilon$  of p))))
  )
))
```

```
abbreviation tangentLineA :: 'a Point set  $\Rightarrow$  'a Point set  $\Rightarrow$  'a Point
 $\Rightarrow$  bool
```

```
where tangentLineA l s x  $\equiv$ 
  (x  $\in$  s)  $\wedge$  (onLine x l)  $\wedge$  (accPoint x s)
 $\wedge$ 
  ( $\forall$  p . ( ((onLine p l)  $\wedge$  (p  $\neq$  x))  $\longrightarrow$ 
    ( $\forall$   $\varepsilon > 0$  .  $\exists$   $\delta > 0$  .  $\forall$  y  $\in$  s . (
      (y within  $\delta$  of x)  $\wedge$  (y  $\neq$  x) )
     $\longrightarrow$ 
    ( $\exists$  r . ((onLine r (lineJoining x y))  $\wedge$  (r within  $\varepsilon$  of p))))
  )
))
```

```
abbreviation hasTangent :: 'a Point set  $\Rightarrow$  'a Point  $\Rightarrow$  bool
```

```
where hasTangent s p  $\equiv$   $\exists$  l . tangentLine l s p
```

The instantaneous velocity of a body is defined to be the velocity

of a co-moving body moving along the tangent line (assuming a tangent line exists).

```
fun vel :: 'a Point set  $\Rightarrow$  'a Point  $\Rightarrow$  'a Space  $\Rightarrow$  bool
  where vel wl p v = (  $\exists$  l . ( tangentLine l wl p )  $\wedge$  ( v  $\in$  lineVelocity l ) )
```

```
lemma lemTangentLineTranslation:
  assumes translation T
  and      tangentLine l s x
  shows     tangentLine (applyToSet (asFunc T) l)
                  (applyToSet (asFunc T) s) (T x)
   $\langle$ proof $\rangle$ 
```

```
lemma lemTangentLineA:
  assumes tangentLine l s x
  shows   tangentLineA l s x
   $\langle$ proof $\rangle$ 
```

```
lemma lemTangentLineE:
  assumes tangentLineA l s x
  and       $\exists$  p  $\neq$  x . onLine p l
  shows     tangentLine l s x
   $\langle$ proof $\rangle$ 
```

end

end

9 Cones

This theory defines (light)cones, regular cones, and their properties.

```
theory Cones
  imports WorldLine TangentLines
begin
```

```
class Cones = WorldLine + TangentLines
begin
```

abbreviation $tl :: 'a \text{ Point set} \Rightarrow \text{Body} \Rightarrow \text{Body} \Rightarrow 'a \text{ Point} \Rightarrow \text{bool}$
where $tl \ l \ m \ b \ x \equiv tangentLine \ l \ (wline \ m \ b) \ x$

The cone of a body at a point comprises the set of points that lie on tangent lines of photons emitted by the body at that point.

abbreviation $cone :: \text{Body} \Rightarrow 'a \text{ Point} \Rightarrow 'a \text{ Point} \Rightarrow \text{bool}$
where $cone \ m \ x \ p$
 $\equiv \exists \ l . (onLine \ p \ l) \wedge (onLine \ x \ l) \wedge (\exists \ ph . Ph \ ph \wedge tl \ l \ m \ ph \ x)$

abbreviation $regularCone :: 'a \text{ Point} \Rightarrow 'a \text{ Point} \Rightarrow \text{bool}$
where $regularCone \ x \ p \equiv \exists \ l . (onLine \ p \ l) \wedge (onLine \ x \ l)$
 $\wedge (\exists \ v \in lineVelocity \ l . sNorm2 \ v = 1)$

abbreviation $coneSet :: \text{Body} \Rightarrow 'a \text{ Point} \Rightarrow 'a \text{ Point set}$
where $coneSet \ m \ x \equiv \{ p . cone \ m \ x \ p \}$

abbreviation $regularConeSet :: 'a \text{ Point} \Rightarrow 'a \text{ Point set}$
where $regularConeSet \ x \equiv \{ p . regularCone \ x \ p \}$

end

end

10 AXIOM: AxLightMinus

This theory declares the axiom AxLightMinus.

theory *AxLightMinus*
imports *WorldLine TangentLines*
begin

AxLightMinus: If an observer sends out a light signal, then the speed of the light signal is 1 according to the observer. Moreover it is possible to send out a light signal in any direction.

class *axLightMinus* = *WorldLine* + *TangentLines*
begin

The definition of AxLightMinus used in this Isabelle proof is slightly different to the one used in the paper-based proof on which it is based. We have established elsewhere, however, that each entails the other in all relevant contexts.

abbreviation $axLightMinusOLD :: Body \Rightarrow 'a \text{ Point} \Rightarrow 'a \text{ Space} \Rightarrow bool$

where $axLightMinusOLD \ m \ p \ v \equiv (m \text{ sees } m \text{ at } p) \longrightarrow ($
 $(\exists \ ph \ . \ (Ph \ ph \wedge (vel \ (wline \ m \ ph) \ p \ v))) \longleftrightarrow (sNorm2 \ v = 1)$
 $)$

abbreviation $axLightMinus :: Body \Rightarrow 'a \text{ Point} \Rightarrow 'a \text{ Space} \Rightarrow bool$

where $axLightMinus \ m \ p \ v \equiv (m \text{ sees } m \text{ at } p)$
 $\longrightarrow (\ \forall \ l \ . \ \forall \ v \in lineVelocity \ l \ .$
 $(\exists \ ph \ . \ (Ph \ ph \wedge (tangentLine \ l \ (wline \ m \ ph) \ p))) \longleftrightarrow$
 $(sNorm2 \ v = 1))$

end

class $AxLightMinus = axLightMinus +$

assumes $AxLightMinus: \forall \ m \ p \ v \ . \ axLightMinus \ m \ p \ v$

begin

end

end

11 Proposition1

This theory shows that observers consider their own lightcones to be upright.

theory $Proposition1$

imports $Cones \ AxLightMinus$

begin

class $Proposition1 = Cones + AxLightMinus$

begin

lemma $lemProposition1:$

assumes $x \in wline \ m \ m$

shows $cone \ m \ x \ p = regularCone \ x \ p$

$\langle proof \rangle$

end

end

12 AXIOM: AxEField

This theory defines the axiom AxEField, which states that the linearly ordered field of quantities is Euclidean, i.e. that all non-negative values have square roots in the field.

```
theory AxEField
  imports Sorts
begin

class axEField = Quantities
begin
  abbreviation axEField :: 'a  $\Rightarrow$  bool
    where axEField x  $\equiv (x \geq 0) \longrightarrow \text{hasRoot } x$ 
end

class AxEField = axEField +
  assumes AxEField:  $\forall x . \text{axEField } x$ 
begin
end

end
```

13 Norms

This theory defines norms, assuming that roots exist.

```
theory Norms
  imports Points AxEField
begin

class Norms = Points + AxEField
begin

abbreviation norm :: 'a Point  $\Rightarrow$  'a ( $\lrcorner$  -  $\lrcorner$ )
  where norm p  $\equiv \text{sqrt } (\text{norm2 } p)$ 

abbreviation sNorm :: 'a Space  $\Rightarrow$  'a
  where sNorm p  $\equiv \text{sqrt } (\text{sNorm2 } p)$ 
```

13.1 axTriangleInequality

Given that norms exist, we can define the triangle inequality for specific cases. This will be asserted more generally as an axiom later.

abbreviation *axTriangleInequality* :: 'a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool
where *axTriangleInequality* *p q* $\equiv$ (*norm* (*p* $\oplus$ *q*) $\leq$ *norm* *p* + *norm* *q*)
 $\langle$ *proof* $\rangle$

lemma *lemNormSqrIsNorm2*: *norm2* *p* = *sqr* (*norm* *p*)
 $\langle$ *proof* $\rangle$

lemma *lemZeroNorm*:
shows (*p* = *origin*) $\longleftrightarrow$ (*norm* *p* = 0)
 $\langle$ *proof* $\rangle$

lemma *lemNormNonNegative*: *norm* *p* $\geq$ 0
 $\langle$ *proof* $\rangle$

lemma *lemNotOriginImpliesPositiveNorm*:
assumes *p* $\neq$ *origin*
shows (*norm* *p* > 0)
 $\langle$ *proof* $\rangle$

lemma *lemNormSymmetry*: *norm* (*p* $\ominus$ *q*) = *norm* (*q* $\ominus$ *p*)
 $\langle$ *proof* $\rangle$

lemma *lemNormOfScaled*: *norm* ($\alpha \otimes p$) = (*abs* α) * (*norm* *p*)
 $\langle$ *proof* $\rangle$

lemma *lemDistancesAdd*:
assumes *triangle*: *axTriangleInequality* (*q* $\ominus$ *p*) (*r* $\ominus$ *q*)
and *distances*: (*x* > 0) $\wedge$ (*y* > 0) $\wedge$ (*sep2* *p* *q* < *sqr* *x*) $\wedge$ (*sep2* *r* *q* < *sqr* *y*)
shows *r* within (*x*+*y*) of *p*
 $\langle$ *proof* $\rangle$

lemma *lemDistancesAddStrictR*:

```

    assumes triangle: axTriangleInequality ( $q \ominus p$ ) ( $r \ominus q$ )
and      distances: (x > 0)  $\wedge$  (y > 0)  $\wedge$  (sep2 p q  $\leq$  sqr x)  $\wedge$  (sep2
r q < sqr y)
    shows r within (x+y) of p
    <proof>

```

```

end

```

```

end

```

14 AxTriangleInequality

This theory declares the Triangle Inequality as an axiom.

```

theory AxTriangleInequality
  imports Norms
begin

```

Although AxTriangleInequality can be proven rather than asserted we have left it as an axiom to illustrate the flexibility of using Isabelle for mathematical physics: well-known mathematical results can be asserted, leaving the researcher free to concentrate on the physics. We can return later to prove the mathematical results when time permits.

```

class AxTriangleInequality = Norms +
  assumes AxTriangleInequality:  $\forall$  p q . axTriangleInequality p q
begin
end

```

```

end

```

15 Sublemma3

This theory establishes how closely tangent lines approximate world lines.

```

theory Sublemma3
  imports WorldLine AxTriangleInequality TangentLines
begin

```

```

class Sublemma3 = WorldLine + AxTriangleInequality + Tangent-
Lines
begin

```

```

lemma sublemma3:
assumes onLine p l
and      norm2 p = 1
and      tangentLine l wl origin
shows
 $\forall \varepsilon > 0 . \exists \delta > 0 . \forall y \text{ ny} . ($ 
 $((y \text{ within } \delta \text{ of origin}) \wedge (y \neq \text{origin}) \wedge (y \in wl) \wedge (\text{norm } y =$ 
 $ny))$ 
 $\longrightarrow$ 
 $((((1/\text{ny}) \otimes y) \text{ within } \varepsilon \text{ of } p) \vee (((-1/\text{ny}) \otimes y) \text{ within } \varepsilon \text{ of } p))$ 
 $)$ 
 $\langle \text{proof} \rangle$ 

```

```

lemma sublemma3Translation:
assumes onLine p l
and      norm2 (p  $\ominus$  x) = 1
and      tangentLine l wl x
shows  $\forall \varepsilon > 0 . \exists \delta > 0 . \forall y \text{ nyx} .$ 
 $((y \text{ within } \delta \text{ of } x) \wedge (y \neq x) \wedge (y \in wl) \wedge (\text{norm } (y \ominus x)$ 
 $= \text{nyx}))$ 
 $\longrightarrow$ 
 $((((1/\text{nyx}) \otimes (y \ominus x)) \text{ within } \varepsilon \text{ of } (p \ominus x))$ 
 $\vee (((-1/\text{nyx}) \otimes (y \ominus x)) \text{ within } \varepsilon \text{ of } (p \ominus x))$ 
 $\langle \text{proof} \rangle$ 

```

end

end

16 Vectors

In this theory we define dot-products, and explain what we mean by timelike, lightlike (null), causal and spacelike vectors.

```

theory Vectors
imports Norms
begin

```

```

class Vectors = Norms
begin

```

```

fun dot :: 'a Point  $\Rightarrow$  'a Point  $\Rightarrow$  'a ( $\langle \cdot \odot \cdot \rangle$ )
where dot u v = (tval u)*(tval v) + (xval u)*(xval v) +
(yval u)*(yval v) + (zval u)*(zval v)

```

fun *sdot* :: 'a Space $\Rightarrow$ 'a Space $\Rightarrow$ 'a ($\lrcorner \odot s \rhd$)
where *sdot* *u v* = (*svalx* *u*)*(*svalx* *v*) + (*svaly* *u*)*(*svaly* *v*) + (*svalz* *u*)*(*svalz* *v*)

fun *mdot* :: 'a Point $\Rightarrow$ 'a Point $\Rightarrow$ 'a ($\lrcorner \odot m \rhd$)
where *mdot* *u v* = (*tval* *u*)*(*tval* *v*) - ((*sComponent* *u*) $\odot s$ (*sComponent* *v*))

abbreviation *timelike* :: 'a Point $\Rightarrow$ bool
where *timelike* *p* $\equiv$ *mNorm2* *p* > 0

abbreviation *lightlike* :: 'a Point $\Rightarrow$ bool
where *lightlike* *p* $\equiv$ (*p* $\neq$ *origin* $\wedge$ *mNorm2* *p* = 0)

abbreviation *spacelike* :: 'a Point $\Rightarrow$ bool
where *spacelike* *p* $\equiv$ *mNorm2* *p* < 0

abbreviation *causal* :: 'a Point $\Rightarrow$ bool
where *causal* *p* $\equiv$ *timelike* *p* $\vee$ *lightlike* *p*

abbreviation *orthog* :: 'a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool
where *orthog* *p q* $\equiv$ (*p* $\odot$ *q*) = 0

abbreviation *orthogs* :: 'a Space $\Rightarrow$ 'a Space $\Rightarrow$ bool
where *orthogs* *p q* $\equiv$ (*p* $\odot s$ *q*) = 0

abbreviation *orthogm* :: 'a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool
where *orthogm* *p q* $\equiv$ (*p* $\odot m$ *q*) = 0

lemma *lemDotDecomposition*:
shows (*u* $\odot$ *v*) = (*tval* *u* * *tval* *v*) + ((*sComponent* *u*) $\odot s$ (*sComponent* *v*))
 $\langle$ *proof* $\rangle$

lemma *lemDotCommute*: *dot* *u v* = *dot* *v u*
 $\langle$ *proof* $\rangle$

lemma *lemDotScaleLeft*: *dot* (*a* $\otimes$ *u*) *v* = *a* * (*dot* *u v*)
 $\langle$ *proof* $\rangle$

lemma *lemDotScaleRight*: *dot* *u* (*a* $\otimes$ *v*) = *a* * (*dot* *u v*)
 $\langle$ *proof* $\rangle$

lemma *lemDotSumLeft*: $\text{dot } (u \oplus v) \ w = (\text{dot } u \ w) + (\text{dot } v \ w)$
 $\langle \text{proof} \rangle$

lemma *lemDotSumRight*: $\text{dot } u \ (v \oplus w) = (\text{dot } u \ v) + (\text{dot } u \ w)$
 $\langle \text{proof} \rangle$

lemma *lemDotDiffLeft*: $\text{dot } (u \ominus v) \ w = (\text{dot } u \ w) - (\text{dot } v \ w)$
 $\langle \text{proof} \rangle$

lemma *lemDotDiffRight*: $\text{dot } u \ (v \ominus w) = (\text{dot } u \ v) - (\text{dot } u \ w)$
 $\langle \text{proof} \rangle$

lemma *lemNorm2OfSum*: $\text{norm2 } (u \oplus v) = \text{norm2 } u + 2 * (u \odot v) + \text{norm2 } v$
 $\langle \text{proof} \rangle$

lemma *lemSDotCommute*: $\text{sdot } u \ v = \text{sdot } v \ u$
 $\langle \text{proof} \rangle$

lemma *lemSDotScaleLeft*: $\text{sdot } (a \otimes s \ u) \ v = a * (\text{sdot } u \ v)$
 $\langle \text{proof} \rangle$

lemma *lemSDotScaleRight*: $\text{sdot } u \ (a \otimes s \ v) = a * (\text{sdot } u \ v)$
 $\langle \text{proof} \rangle$

lemma *lemSDotSumLeft*: $\text{sdot } (u \oplus s \ v) \ w = (\text{sdot } u \ w) + (\text{sdot } v \ w)$
 $\langle \text{proof} \rangle$

lemma *lemSDotSumRight*: $\text{sdot } u \ (v \oplus s \ w) = (\text{sdot } u \ v) + (\text{sdot } u \ w)$
 $\langle \text{proof} \rangle$

lemma *lemSDotDiffLeft*: $\text{sdot } (u \ominus s \ v) \ w = (\text{sdot } u \ w) - (\text{sdot } v \ w)$
 $\langle \text{proof} \rangle$

lemma *lemSDotDiffRight*: $\text{sdot } u \ (v \ominus s \ w) = (\text{sdot } u \ v) - (\text{sdot } u \ w)$
 $\langle \text{proof} \rangle$

lemma *lemMDotDiffLeft*: $\text{mdot } (u \ominus v) \ w = (\text{mdot } u \ w) - (\text{mdot } v \ w)$
 $\langle \text{proof} \rangle$

lemma *lemMDotSumLeft*: $\text{mdot } (u \oplus v) \ w = (\text{mdot } u \ w) + (\text{mdot } v \ w)$
 $\langle \text{proof} \rangle$

lemma *lemMDotScaleLeft*: $\text{mdot } (a \otimes u) v = a * (\text{mdot } u v)$
 $\langle \text{proof} \rangle$

lemma *lemMDotScaleRight*: $\text{mdot } u (a \otimes v) = a * (\text{mdot } u v)$
 $\langle \text{proof} \rangle$

lemma *lemSNorm2OfSum*: $s\text{Norm2 } (u \oplus s v) = s\text{Norm2 } u + 2*(u \odot s v) + s\text{Norm2 } v$
 $\langle \text{proof} \rangle$

lemma *lemSNormNonNeg*:
shows $s\text{Norm } v \geq 0$
 $\langle \text{proof} \rangle$

lemma *lemMNorm2OfSum*: $m\text{Norm2 } (u \oplus v) = m\text{Norm2 } u + 2*(u \odot m v) + m\text{Norm2 } v$
 $\langle \text{proof} \rangle$

lemma *lemMNorm2OfDiff*: $m\text{Norm2 } (u \ominus v) = m\text{Norm2 } u - 2*(u \odot m v) + m\text{Norm2 } v$
 $\langle \text{proof} \rangle$

lemma *lemMNorm2Decomposition*: $m\text{Norm2 } p = (p \odot m p)$
 $\langle \text{proof} \rangle$

lemma *lemMDecomposition*:
assumes $(u \odot m v) \neq 0$
and $m\text{Norm2 } v \neq 0$
and $a = (u \odot m v) / (m\text{Norm2 } v)$
and $up = (a \otimes v)$
and $uo = (u \ominus up)$
shows $u = (up \oplus uo) \wedge \text{parallel } up v \wedge \text{orthogm } uo v \wedge (up \odot m v) = (u \odot m v)$
 $\langle \text{proof} \rangle$

end

end

17 CauchySchwarz

This theory defines and proves the Cauchy-Schwarz inequality for both spatial and spacetime vectors.

theory *CauchySchwarz*
 imports *Vectors*
begin

We essentially prove the same result twice, once for 3-dimensional spatial points, and once for 4-dimensional spacetime points. While this is clearly inefficient, it keeps things straightforward for non-Isabelle experts.

class *CauchySchwarz* = *Vectors*
begin

lemma *lemCauchySchwarz4*:
 shows $\text{abs } (\text{dot } u \ v) \leq (\text{norm } u) * (\text{norm } v)$
 $\langle \text{proof} \rangle$

lemma *lemCauchySchwarzSqr4*:
 shows $\text{sqr}(\text{dot } u \ v) \leq (\text{norm2 } u) * (\text{norm2 } v)$
 $\langle \text{proof} \rangle$

lemma *lemCauchySchwarz*:
 shows $\text{abs } (\text{sdot } u \ v) \leq (\text{sNorm } u) * (\text{sNorm } v)$
 $\langle \text{proof} \rangle$

lemma *lemCauchySchwarzSqr*:
 shows $\text{sqr}(\text{sdot } u \ v) \leq (\text{sNorm2 } u) * (\text{sNorm2 } v)$
 $\langle \text{proof} \rangle$

lemma *lemCauchySchwarzEquality*:
 assumes $\text{sqr } (\text{sdot } u \ v) = (\text{sNorm2 } u) * (\text{sNorm2 } v)$
 and $u \neq \text{sOrigin} \wedge v \neq \text{sOrigin}$

```

shows    $\exists a \neq 0 . u = (a \otimes s v)$ 
 $\langle proof \rangle$ 

```

```

lemma lemCauchySchwarzEqualityInUnitSphere:
  assumes  $(sNorm2\ u \leq 1) \wedge (sNorm2\ v \leq 1)$ 
and       $sdot\ u\ v = 1$ 
shows     $u = v$ 
 $\langle proof \rangle$ 

```

```

lemma lemCausalOrthogonToLightlikeImpliesParallel:
  assumes causal  $p$ 
and     lightlike  $q$ 
and     orthogon  $p\ q$ 
shows    parallel  $p\ q$ 
 $\langle proof \rangle$ 

```

```

end

```

```

end

```

18 Matrices

This theory defines 4×4 matrices.

```

theory Matrices
  imports Vectors
begin

```

```

record 'a Matrix =
  trow :: 'a Point
  xrow :: 'a Point
  yrow :: 'a Point
  zrow :: 'a Point

```

```

class Matrices = Vectors
begin

```

```

fun applyMatrix :: 'a Matrix  $\Rightarrow$  'a Point  $\Rightarrow$  'a Point
  where applyMatrix  $m\ p = \langle$   $tval = dot\ (trow\ m)\ p,$   $xval = dot\ (xrow$ 
 $m)\ p,$ 
 $yval = dot\ (yrow\ m)\ p,$   $zval = dot\ (zrow\ m)\ p \rangle$ 

```

```

fun tcol :: 'a Matrix  $\Rightarrow$  'a Point
  where tcol  $m = \langle$   $tval = tval\ (trow\ m),$   $xval = tval\ (xrow\ m),$ 

```

```

      yval = tval (yrow m), zval = tval (zrow m) )

fun xcol :: 'a Matrix ⇒ 'a Point
  where xcol m = (| tval = xval (trow m), xval = xval (xrow m),
                  yval = xval (yrow m), zval = xval (zrow m) |)

fun ycol :: 'a Matrix ⇒ 'a Point
  where ycol m = (| tval = yval (trow m), xval = yval (xrow m),
                  yval = yval (yrow m), zval = yval (zrow m) |)

fun zcol :: 'a Matrix ⇒ 'a Point
  where zcol m = (| tval = zval (trow m), xval = zval (xrow m),
                  yval = zval (yrow m), zval = zval (zrow m) |)

fun transpose :: 'a Matrix ⇒ 'a Matrix
  where transpose m = (| trow = (tcol m), xrow = (xcol m),
                       yrow = (ycol m), zrow = (zcol m) |)

fun mprod :: 'a Matrix ⇒ 'a Matrix ⇒ 'a Matrix
  where mprod m1 m2 =
    transpose (| trow = applyMatrix m1 (tcol m2), xrow =
                applyMatrix m1 (xcol m2),
                yrow = applyMatrix m1 (ycol m2), zrow =
                applyMatrix m1 (zcol m2) |)

end

end

```

19 LinearMaps

This theory defines linear maps and establishes their main properties.

```

theory LinearMaps
  imports Functions CauchySchwarz Matrices
begin

```

```

class LinearMaps = Functions + CauchySchwarz + Matrices
begin

```

```

abbreviation linear :: ('a Point ⇒ 'a Point) ⇒ bool where

```

$linear\ L \equiv (L\ origin = origin)$
 $\wedge (\forall\ a\ p .\ L\ (a \otimes p) = (a \otimes (L\ p)))$
 $\wedge (\forall\ p\ q .\ L\ (p \oplus q) = ((L\ p) \oplus (L\ q)))$
 $\wedge (\forall\ p\ q .\ L\ (p \ominus q) = ((L\ p) \ominus (L\ q)))$

lemma *lemLinearProps:*
assumes *linear L*
shows $(L\ origin = origin) \wedge (L\ (a \otimes p) = (a \otimes (L\ p)))$
 $\wedge (L\ (p \oplus q) = ((L\ p) \oplus (L\ q)))$
 $\wedge (L\ (p \ominus q) = ((L\ p) \ominus (L\ q)))$
 $\langle proof \rangle$

lemma *lemMatrixApplicationIsLinear:* *linear (applyMatrix m)*
 $\langle proof \rangle$

lemma *lemLinearIsMatrixApplication:*
assumes *linear L*
shows $\exists\ m .\ L = (applyMatrix\ m)$
 $\langle proof \rangle$

lemma *lemLinearIffMatrix:* *linear L* $\longleftrightarrow (\exists\ M .\ L = applyMatrix\ M)$
 $\langle proof \rangle$

lemma *lemIdIsLinear:* *linear id*
 $\langle proof \rangle$

lemma *lemLinearIsBounded:*
assumes *linear L*
shows *bounded L*
 $\langle proof \rangle$

lemma *lemLinearIsCts:*

```

assumes linear L
shows   cts (asFunc L) x
<proof>

```

```

lemma lemLinOfLinIsLin:
assumes (linear A)  $\wedge$  (linear B)
shows   linear (B  $\circ$  A)
<proof>

```

```

lemma lemInverseLinear:
  assumes linear A
and      invertible A
shows     $\exists A' . (linear\ A') \wedge (\forall\ p\ q. A\ p = q \longleftrightarrow A'\ q = p)$ 
<proof>

```

end

end

20 Affine

This theory defines affine transformations and established their key properties.

```

theory Affine
  imports Translations LinearMaps
begin

```

```

class Affine = Translations + LinearMaps
begin

```

```

abbreviation affine :: ('a Point  $\Rightarrow$  'a Point)  $\Rightarrow$  bool
  where affine A  $\equiv \exists\ L\ T . (linear\ L) \wedge (translation\ T) \wedge (A = T \circ L)$ 

```

```

abbreviation affInvertible :: ('a Point  $\Rightarrow$  'a Point)  $\Rightarrow$  bool
  where affInvertible A  $\equiv affine\ A \wedge invertible\ A$ 

```

abbreviation $isLinearPart :: ('a \text{ Point} \Rightarrow 'a \text{ Point}) \Rightarrow ('a \text{ Point} \Rightarrow 'a \text{ Point}) \Rightarrow bool$

where $isLinearPart \ A \ L \equiv (affine \ A) \wedge (linear \ L) \wedge$
 $(\exists \ T. (translation \ T \wedge A = T \circ L))$

abbreviation $isTranslationPart :: ('a \text{ Point} \Rightarrow 'a \text{ Point}) \Rightarrow ('a \text{ Point} \Rightarrow 'a \text{ Point}) \Rightarrow bool$

where $isTranslationPart \ A \ T \equiv (affine \ A) \wedge (translation \ T) \wedge$
 $(\exists \ L. (linear \ L \wedge A = T \circ L))$

20.1 Affine approximation

A key concept in the proof is affine approximation. We will eventually assert that worldview transformation can be approximated by invertible affine transformations.

abbreviation $affineApprox :: ('a \text{ Point} \Rightarrow 'a \text{ Point}) \Rightarrow$
 $(('a \text{ Point} \Rightarrow 'a \text{ Point} \Rightarrow bool) \Rightarrow$
 $'a \text{ Point} \Rightarrow bool)$

where $affineApprox \ A \ f \ x \equiv (isFunction \ f) \wedge$
 $(affInvertible \ A) \wedge (diffApprox \ (asFunc \ A) \ f \ x)$

fun $applyAffineToLine :: ('a \text{ Point} \Rightarrow 'a \text{ Point})$
 $\Rightarrow 'a \text{ Point set} \Rightarrow 'a \text{ Point set} \Rightarrow bool$
where $applyAffineToLine \ A \ l \ l' \longleftrightarrow (affine \ A) \wedge$
 $(\exists \ T \ L \ b \ d. ((linear \ L) \wedge (translation \ T) \wedge (A = T \circ L) \wedge$
 $(l = line \ b \ d) \wedge (l' = (line \ (A \ b) \ (L \ d))))))$

abbreviation $affConstantOn :: ('a \text{ Point} \Rightarrow 'a \text{ Point}) \Rightarrow 'a \text{ Point} \Rightarrow$
 $'a \text{ Point set} \Rightarrow bool$

where $affConstantOn \ A \ x \ s \equiv (\exists \ \varepsilon > 0. \forall \ y \in s. (y \text{ within } \varepsilon \text{ of } x) \longrightarrow$
 $((A \ y) = (A \ x)))$

lemma $lemTranslationPartIsUnique:$

assumes $isTranslationPart \ A \ T1$

and $isTranslationPart \ A \ T2$

shows $T1 = T2$

$\langle proof \rangle$

lemma $lemLinearPartIsUnique:$

assumes *isLinearPart A L1*
and *isLinearPart A L2*
shows $L1 = L2$
 $\langle proof \rangle$

lemma *lemLinearImpliesAffine:*
assumes *linear L*
shows *affine L*
 $\langle proof \rangle$

lemma *lemTranslationImpliesAffine:*
assumes *translation T*
shows *affine T*
 $\langle proof \rangle$

lemma *lemAffineDiff:*
assumes *linear L*
and $\exists T. ((translation\ T) \wedge (A = T \circ L))$
shows $((A\ p) \ominus (A\ q)) = L\ (p \ominus q)$
 $\langle proof \rangle$

lemma *lemAffineImpliesTotalFunction:*
assumes *affine A*
shows *isTotalFunction (asFunc A)*
 $\langle proof \rangle$

lemma *lemAffineEqualAtBase:*
assumes *affineApprox A f x*
shows $\forall y. (f\ x\ y) \longleftrightarrow (y = A\ x)$
 $\langle proof \rangle$

lemma *lemAffineOfPointOnLine:*
assumes $(linear\ L) \wedge (translation\ T) \wedge (A = T \circ L)$
and $x = (b \oplus (a \otimes d))$
shows $A\ x = ((A\ b) \oplus (a \otimes (L\ d)))$
 $\langle proof \rangle$

lemma *lemAffineOfLineIsLine:*
assumes *isLine l*
shows $(applyAffineToLine\ A\ l\ l') \longleftrightarrow (affine\ A \wedge l' = applyToSet$

(asFunc A) l)
 ⟨proof⟩

lemma *lemOnLineUnderAffine*:
 assumes (affine A) ∧ (onLine p l)
 shows onLine (A p) (applyToSet (asFunc A) l)
 ⟨proof⟩

lemma *lemLineJoiningUnderAffine*:
 assumes affine A
 shows applyToSet (asFunc A) (lineJoining p q) = lineJoining (A p) (A q)
 ⟨proof⟩

lemma *lemAffineIsCts*:
 assumes affine A
 shows cts (asFunc A) x
 ⟨proof⟩

lemma *lemAffineContinuity*:
 assumes affine A
 shows $\forall x. \forall \varepsilon > 0. \exists \delta > 0. \forall p. (p \text{ within } \delta \text{ of } x) \longrightarrow ((A p) \text{ within } \varepsilon \text{ of } (A x))$
 ⟨proof⟩

lemma *lemAffOfAffIsAff*:
 assumes (affine A) ∧ (affine B)
 shows affine (B ∘ A)
 ⟨proof⟩

lemma *lemInverseAffine*:
 assumes affInvertible A
 shows $\exists A' . (affine A') \wedge (\forall p q . A p = q \longleftrightarrow A' q = p)$
 ⟨proof⟩

lemma *lemAffineApproxDomainTranslation*:

```

assumes translation T
and      affineApprox A f x
and       $\forall p\ q. T\ p = q \longleftrightarrow T'\ q = p$ 
shows    affineApprox (A ∘ T) (composeRel f (asFunc T)) (T' x)
<proof>

```

```

lemma lemAffineApproxRangeTranslation:
  assumes translation T
and      affineApprox A f x
shows    affineApprox (T ∘ A) (composeRel (asFunc T) f) x
<proof>

```

```

lemma lemAffineIdentity:
  assumes affine A
and       $e > 0$ 
and       $\forall y. (y \text{ within } e \text{ of } x) \longrightarrow (A\ y = y)$ 
shows     $A = id$ 
<proof>

```

end

end

21 Sublemma4

This theory shows that functions with affine approximations are continuous where approximated.

```

theory Sublemma4
  imports Affine AxTriangleInequality
begin

```

Our naming of lemmas, propositions, etc., is sometimes counterintuitive. This is because the proof follows a hand-written proof, and we need to maintain the link between the paper-based and Isabelle versions. We will specifically be discussing how we translated from one to the other in a forthcoming paper (under construction). In fact, sublemmas 1 and 2 were eventually found to be unnecessary during construction of the Isabelle proof, and so do not appear in this documentation.

```

class Sublemma4 = Affine + AxTriangleInequality

```

```

begin

lemma sublemma4:
  assumes affineApprox A f x
  shows ( $\exists \delta > 0. \forall p. (p \text{ within } \delta \text{ of } x) \longrightarrow (definedAt f p) \wedge (cts f x)$ )
  <proof>

end

end

```

22 MainLemma

This theory establishes conditions under which a function maps tangent lines to tangent lines.

```

theory MainLemma
  imports Sublemma3 Sublemma4
begin

class MainLemma = Sublemma3 + Sublemma4
begin

lemma lemMainLemmaBasic:
  assumes tgt:    tangentLine l wl origin
and    injf:    injective f
and    affapp:   affineApprox A f origin
and    f00:     f origin origin
and    ctsf'0:   cts (invFunc f) origin
and    affline:  applyAffineToLine A l l'
shows    tangentLine l' (applyToSet f wl) origin
  <proof>

lemma lemMainLemmaOrigin:
  assumes tgtx:   tangentLine l wl x
and    injf:    injective f
and    affappx:  affineApprox A f x
and    fx0:     f x origin
and    ctsf'0:   cts (invFunc f) origin
and    affline:  applyAffineToLine A l l'
shows    tangentLine l' (applyToSet f wl) origin

```

$\langle proof \rangle$

```

lemma lemMainLemma:
  assumes tgtx:    tangentLine l wl x
and      injf:    injective f
and      affappx: affineApprox A f x
and      fxy:     f x y
and      ctsf'y:   cts (invFunc f) y
and      affline: applyAffineToLine A l l'
shows    tangentLine l' (applyToSet f wl) y
 $\langle proof \rangle$ 

```

end

end

23 AXIOM: AxDiff

This theory declares the axiom AxDiff.

```

theory AxDiff
  imports Affine WorldView
begin

  AxDiff: Worldview transformations are differentiable wherever
  they are defined - they can be approximated locally by affine
  transformations.

  class axDiff = Affine + WorldView
  begin
    abbreviation axDiff :: Body  $\Rightarrow$  Body  $\Rightarrow$  'a Point  $\Rightarrow$  bool
      where axDiff m k p  $\equiv$  (definedAt (wvtFunc m k) p)
         $\longrightarrow$  ( $\exists$  A . (affineApprox A (wvtFunc m k) p))
  end

```

```

class AxDiff = axDiff +
  assumes AxDiff:  $\forall$  m k p . axDiff m k p
begin
end

end

```

24 TangentLineLemma

This theory shows that affine approximations map tangent lines to tangent lines.

```
theory TangentLineLemma
imports MainLemma AxDiff Cones
begin
```

```
class TangentLineLemma = MainLemma + AxDiff + Cones
begin
```

```
lemma lemWVTImpliesFunction: isFunction (wvtFunc k h)
  <proof>
```

```
lemma lemWVTCts:
  assumes definedAt (wvtFunc h k) p
  shows cts (wvtFunc h k) p
  <proof>
```

```
lemma lemWVTInverse: invFunc (wvtFunc k h) = wvtFunc h k
  <proof>
```

```
lemma lemWVTInverseCts:
  assumes wvtFunc k h p q
  shows cts (wvtFunc h k) q
  <proof>
```

```
lemma lemWVTInjective: injective (wvtFunc k h)
  <proof>
```

```
lemma lemPresentation:
  assumes x ∈ wline m b
and tangentLine l (wline m b) x
and affineApprox A (wvtFunc m k) x
and wvtFunc m k x y
and applyAffineToLine A l l'
shows tangentLine l' (wline k b) y
  <proof>
```

```

lemma lemTangentLines:
  assumes affineApprox A (wvtFunc m k) x
  and      tl l m b x
  and      applyAffineToLine A l l'
  and      wvtFunc m k x y
  shows    tl l' k b y
  <proof>

```

```

lemma lemSelfTangentIsTimeAxis:
  assumes tangentLine l (wline k k) x
  shows    l = timeAxis
  <proof>

```

```

lemma lemTangentLineUnique:
  assumes tl l1 m k x
  and      tl l2 m k x
  and      affineApprox A (wvtFunc m k) x
  and      wvtFunc m k x y
  and      x ∈ wline m k
  shows    l1 = l2
  <proof>

```

end

end

25 Proposition2

This theory shows that affine approximations map surfaces of cones to (subsets of) surfaces of cones.

```

theory Proposition2
  imports TangentLineLemma
begin

```

```

class Proposition2 = TangentLineLemma
begin

```

```

lemma lemProposition2:
  assumes affineApprox A (wvtFunc m k) x
  shows    applyToSet (asFunc A) (coneSet m x) ⊆ coneSet k (A x)

```

$\langle proof \rangle$

end

end

26 AXIOM: AxEventMinus

This theory declares the axiom AxEventMinus

```
theory AxEventMinus
  imports WorldView
begin
```

AxEventMinus: An observer encounters the events in which they are observed.

```
class axEventMinus = WorldView
begin
```

```
  abbreviation axEventMinus :: Body  $\Rightarrow$  Body  $\Rightarrow$  'a Point  $\Rightarrow$  bool
    where axEventMinus m k p  $\equiv$  (m sees k at p)
       $\longrightarrow (\exists q . \forall b . (m \text{ sees } b \text{ at } p) \longleftrightarrow (k \text{ sees } b \text{ at } q))$ 
```

end

```
class AxEventMinus = axEventMinus +
  assumes AxEventMinus:  $\forall m k p . axEventMinus m k p$ 
begin
end
```

end

27 Proposition3

This theory collects together earlier results to show that world-view transformations can be approximated by affine transformations that have various useful properties.

```
theory Proposition3
  imports Proposition1 Proposition2 AxEventMinus
begin
```

```

class Proposition3 = Proposition1 + Proposition2 + AxEventMinus
begin

```

```

lemma lemProposition3:
  assumes m sees k at x
  shows  $\exists A y . (wvtFunc\ m\ k\ x\ y)$ 
         $\wedge$   $(affineApprox\ A\ (wvtFunc\ m\ k)\ x)$ 
         $\wedge$   $(applyToSet\ (asFunc\ A)\ (coneSet\ m\ x) \subseteq coneSet\ k\ y)$ 
         $\wedge$   $(coneSet\ k\ y = regularConeSet\ y)$ 
   $\langle proof \rangle$ 

```

```

end

```

```

end

```

28 ObserverConeLemma

This theory gives sufficient conditions for an observed observer's cone to appear upright to that observer.

```

theory ObserverConeLemma
  imports Proposition3
begin

```

```

class ObserverConeLemma = Proposition3
begin

```

```

lemma lemConeOfObserved:
  assumes affineApprox A (wvtFunc m k) x
  and m sees k at x
  shows  $coneSet\ k\ (A\ x) = regularConeSet\ (A\ x)$ 
   $\langle proof \rangle$ 

```

```

end

```

```

end

```

29 Quadratics

This theory shows how to find the roots of a quadratic, assuming that roots exist (AxEField).

```

theory Quadratics
  imports Functions AxEField
begin

class Quadratics = Functions + AxEField
begin

abbreviation quadratic :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  ('a  $\Rightarrow$  'a)
  where quadratic a b c  $\equiv \lambda x . a*(sqr\ x) + b*x + c$ 

abbreviation groot :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  bool
  where groot a b c r  $\equiv$  (quadratic a b c) r = 0

abbreviation roots :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  'a set
  where roots a b c  $\equiv \{ r . groot\ a\ b\ c\ r \}$ 

abbreviation discriminant :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  'a
  where discriminant a b c  $\equiv$  (sqr b) - 4*a*c

abbreviation qcase1 :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  bool
  where qcase1 a b c  $\equiv$  (a = 0  $\wedge$  b = 0  $\wedge$  c = 0)
abbreviation qcase2 :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  bool
  where qcase2 a b c  $\equiv$  (a = 0  $\wedge$  b = 0  $\wedge$  c  $\neq$  0)
abbreviation qcase3 :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  bool
  where qcase3 a b c  $\equiv$  (a = 0  $\wedge$  b  $\neq$  0  $\wedge$  (c = 0  $\vee$  c  $\neq$  0))
abbreviation qcase4 :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  bool
  where qcase4 a b c  $\equiv$  (a  $\neq$  0  $\wedge$  discriminant a b c < 0)
abbreviation qcase5 :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  bool
  where qcase5 a b c  $\equiv$  (a  $\neq$  0  $\wedge$  discriminant a b c = 0)
abbreviation qcase6 :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  bool
  where qcase6 a b c  $\equiv$  (a  $\neq$  0  $\wedge$  discriminant a b c > 0)

lemma lemQuadRootCondition:
  assumes a  $\neq$  0
  shows (sqr (2*a*r + b) = discriminant a b c)  $\longleftrightarrow$  groot a b c r
   $\langle proof \rangle$ 

lemma lemQuadraticCasesComplete:
  shows qcase1 a b c  $\vee$  qcase2 a b c  $\vee$  qcase3 a b c  $\vee$  qcase4 a b c  $\vee$ 
    qcase5 a b c  $\vee$  qcase6 a b c
   $\langle proof \rangle$ 

```

lemma *lemQCase1*:
assumes *qcase1 a b c*
shows $\forall r . \text{qroot } a \ b \ c \ r$
 $\langle \text{proof} \rangle$

lemma *lemQCase2*:
assumes *qcase2 a b c*
shows $\neg (\exists r . \text{qroot } a \ b \ c \ r)$
 $\langle \text{proof} \rangle$

lemma *lemQCase3*:
assumes *qcase3 a b c*
shows $\text{qroot } a \ b \ c \ r \longleftrightarrow r = -c/b$
 $\langle \text{proof} \rangle$

lemma *lemQCase4*:
assumes *qcase4 a b c*
shows $\neg (\exists r . \text{qroot } a \ b \ c \ r)$
 $\langle \text{proof} \rangle$

lemma *lemQCase5*:
assumes *qcase5 a b c*
shows $\text{qroot } a \ b \ c \ r \longleftrightarrow r = -b/(2*a)$
 $\langle \text{proof} \rangle$

lemma *lemQCase6*:
assumes *qcase6 a b c*
and *rd = sqrt (discriminant a b c)*
and *rp = ((-b) + rd) / (2*a)*
and *rm = ((-b) - rd) / (2*a)*
shows $(rp \neq rm) \wedge \text{qroots } a \ b \ c = \{ rp, rm \}$
 $\langle \text{proof} \rangle$

lemma *lemQuadraticRootCount*:
assumes $\neg(\text{qcase1 } a \ b \ c)$
shows $\text{finite } (\text{qroots } a \ b \ c) \wedge \text{card } (\text{qroots } a \ b \ c) \leq 2$
 $\langle \text{proof} \rangle$

end

end

30 Classification

This theory explains how to establish whether a point lies inside, on or outside a cone.

theory *Classification*

imports *Cones Quadratics CauchySchwarz*

begin

We want to establish where a point lies in relation to a cone, and will later show that this relationship is preserved under relevant affine transformations. We therefore need a classification scheme that relies on purely affine concepts. To do this we consider lines that can be drawn through the point, and ask how many points lie in the intersection of such a line and the cone.

class *Classification* = *Cones* + *Quadratics* + *CauchySchwarz*

begin

abbreviation *vertex* :: 'a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool

where *vertex* *x p* $\equiv$ (*x* = *p*)

abbreviation *insideRegularCone* :: 'a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool

where *insideRegularCone* *x p* $\equiv$

(*slopeFinite* *x p*) $\wedge$ ($\exists$ *v* $\in$ *lineVelocity* (*lineJoining* *x p*) . *sNorm2* *v* < 1)

abbreviation *outsideRegularCone* :: 'a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool

where *outsideRegularCone* *x p* $\equiv$

(*x* $\neq$ *p*) $\wedge$
((*slopeInfinite* *x p*) $\vee$ ($\exists$ *v* $\in$ *lineVelocity* (*lineJoining* *x p*) .
sNorm2 *v* > 1))

abbreviation *onRegularCone* :: 'a Point $\Rightarrow$ 'a Point $\Rightarrow$ bool

where *onRegularCone* *x p* $\equiv$ (*x* = *p*) $\vee$ ($\exists$ *v* $\in$ *lineVelocity* (*lineJoining* *x p*) .
sNorm2 *v* = 1)

lemma *lemDrtnLineJoining*:
assumes $l = \text{lineJoining } x \ p$
and $x \neq p$
shows $(p \ominus x) \in \text{drtn } l$
 $\langle \text{proof} \rangle$

lemma *lemVelocityLineJoining*:
assumes $l = \text{lineJoining } x \ p$
and $v = \text{velocityJoining origin } (p \ominus x)$
and $x \neq p$
shows $v \in \text{lineVelocity } l$
 $\langle \text{proof} \rangle$

lemma *lemSlopeLineJoining*:
assumes $l = \text{lineJoining } p \ q$
and $p \neq q$
shows $\text{lineSlopeFinite } l \longleftrightarrow \text{slopeFinite } p \ q$
 $\langle \text{proof} \rangle$

lemma *lemVelocityJoiningUsingPoints*:
assumes $p \neq q$
shows $\text{velocityJoining } p \ q = \text{velocityJoining origin } (q \ominus p)$
 $\langle \text{proof} \rangle$

lemma *lemLineVelocityNonZeroImpliesFinite*:
assumes $u \in \text{lineVelocity } l$
and $s\text{Norm2 } u \neq 0$
shows $\text{lineSlopeFinite } l$
 $\langle \text{proof} \rangle$

lemma *lemLineVelocityUsingPoints*:
assumes $\text{slopeFinite } p \ q$
and $\text{onLine } p \ l \wedge \text{onLine } q \ l$
shows $\text{lineVelocity } l = \{ \text{velocityJoining } p \ q \}$
 $\langle \text{proof} \rangle$

lemma *lemSNorm2VelocityJoining:*
assumes *slopeFinite x p*
and $v = \text{velocityJoining } x \ p$
shows $\text{sqr } (\text{tval } p - \text{tval } x) * \text{sNorm2 } v = \text{sNorm2 } (\text{sComponent } (p \ominus x))$
 $\langle \text{proof} \rangle$

lemma *lemOrthogonalSpaceVectorExists:*
shows $\exists w . (w \neq \text{sOrigin}) \wedge (w \odot_s v) = 0$
 $\langle \text{proof} \rangle$

lemma *lemNonParallelVectorsExist:*
shows $\exists w . ((w \neq \text{origin}) \wedge (\text{tval } v = \text{tval } w)) \wedge (\neg (\exists \alpha . (\alpha \neq 0) \wedge v = (\alpha \otimes w)))$
 $\langle \text{proof} \rangle$

lemma *lemConeContainsVertex:*
shows *regularCone x x*
 $\langle \text{proof} \rangle$

lemma *lemConesExist:*
shows *regularConeSet x $\neq$ {}*
 $\langle \text{proof} \rangle$

lemma *lemRegularCone:*
shows $((x = p) \vee \text{onRegularCone } x \ p) \longleftrightarrow \text{regularCone } x \ p$
 $\langle \text{proof} \rangle$

lemma *lemSlopeInfiniteImpliesOutside:*
assumes $x \neq p$
and *slopeInfinite x p*
shows $\exists l \ p' . (p' \neq p) \wedge \text{onLine } p' \ l \wedge \text{onLine } p \ l$
 $\wedge (l \cap \text{regularConeSet } x = \{\})$
 $\langle \text{proof} \rangle$

lemma *lemClassification:*
shows $(\text{insideRegularCone } x \ p) \vee (\text{vertex } x \ p \vee \text{outsideRegularCone } x \ p \vee \text{onRegularCone } x \ p)$
 $\langle \text{proof} \rangle$

lemma *lemQuadCoordinates:*
assumes $p = (B \oplus (\alpha \otimes D))$
and $a = mNorm2\ D$
and $b = 2*(tval\ (B \ominus x))*(tval\ D) - 2*((sComponent\ D) \odot s\ (sComponent\ (B \ominus x)))$
and $c = mNorm2\ (B \ominus x)$
shows $sqr\ (tval\ (p \ominus x)) - sNorm2\ (sComponent\ (p \ominus x)) = a*(sqr\ \alpha) + b*\alpha + c$
 $\langle proof \rangle$

lemma *lemConeCoordinates:*
shows $(onRegularCone\ x\ p \longleftrightarrow sqr\ (tval\ p - tval\ x) = sNorm2\ (sComponent\ (p \ominus x)))$
 $\wedge\ (insideRegularCone\ x\ p \longleftrightarrow sqr\ (tval\ p - tval\ x) > sNorm2\ (sComponent\ (p \ominus x)))$
 $\wedge\ (outsideRegularCone\ x\ p \longleftrightarrow sqr\ (tval\ p - tval\ x) < sNorm2\ (sComponent\ (p \ominus x)))$
 $\langle proof \rangle$

lemma *lemConeCoordinates1:*
shows $p \in regularConeSet\ x \longleftrightarrow norm2\ (p \ominus x) = 2*sqr\ (tval\ p - tval\ x)$
 $\langle proof \rangle$

lemma *lemWhereLineMeetsCone:*
assumes $a = mNorm2\ D$
and $b = 2*(tval\ (B \ominus x))*(tval\ D) - 2*((sComponent\ D) \odot s\ (sComponent\ (B \ominus x)))$
and $c = mNorm2\ (B \ominus x)$
shows $groot\ a\ b\ c\ \alpha \longleftrightarrow regularCone\ x\ (B \oplus (\alpha \otimes D))$
 $\langle proof \rangle$

lemma *lemLineMeetsCone1:*
assumes $\neg (x \in l)$
and $isLine\ l$
and $S = l \cap regularConeSet\ x$
and $l: l = line\ B\ D$
and $X: X = (B \ominus x)$
and $a: a = mNorm2\ D$
and $b: b = 2*(tval\ X)*(tval\ D) - 2*((sComponent\ D) \odot s\ (sComponent\ X))$
and $c: c = mNorm2\ X$
shows $(qcase1\ a\ b\ c \longrightarrow S = \{B\})$

$\langle proof \rangle$

lemma *lemLineMeetsCone2*:

assumes $\neg (x \in l)$
and $isLine\ l$
and $S = l \cap regularConeSet\ x$
and $l: l = line\ B\ D$
and $X: X = (B \ominus x)$
and $a = mNorm2\ D$
and $b = 2*(tval\ (B \ominus x))*(tval\ D) - 2*((sComponent\ D) \odot s\ (sComponent\ (B \ominus x)))$
and $c = mNorm2\ (B \ominus x)$
shows $qcase2\ a\ b\ c \longrightarrow S = \{\}$
 $\langle proof \rangle$

lemma *lemLineMeetsCone3*:

assumes $\neg (x \in l)$
and $isLine\ l$
and $S = l \cap regularConeSet\ x$
and $l: l = line\ B\ D$
and $X: X = (B \ominus x)$
and $a: a = mNorm2\ D$
and $b: b = 2*(tval\ X)*(tval\ D) - 2*((sComponent\ D) \odot s\ (sComponent\ X))$
and $c: c = sqr\ (tval\ X) - sNorm2\ (sComponent\ X)$
and $y3: y3 = (B \oplus ((-c/b) \otimes D))$
shows $qcase3\ a\ b\ c \longrightarrow S = \{y3\}$
 $\langle proof \rangle$

lemma *lemLineMeetsCone4*:

assumes $\neg (x \in l)$
and $isLine\ l$
and $S = l \cap regularConeSet\ x$
and $l: l = line\ B\ D$
and $X: X = (B \ominus x)$
and $a: a = mNorm2\ D$
and $b: b = 2*(tval\ X)*(tval\ D) - 2*((sComponent\ D) \odot s\ (sComponent\ X))$
and $c: c = sqr\ (tval\ X) - sNorm2\ (sComponent\ X)$
shows $(qcase4\ a\ b\ c \longrightarrow S = \{\})$
 $\langle proof \rangle$

lemma *lemLineMeetsCone5*:
assumes $\neg (x \in l)$
and *isLine* l
and $S = l \cap \text{regularConeSet } x$
and $l: l = \text{line } B \ D$
and $X: X = (B \ominus x)$
and $a: a = \text{mNorm2 } D$
and $b: b = 2*(\text{tval } X)*(\text{tval } D) - 2*((\text{sComponent } D) \odot_s (\text{sComponent } X))$
and $c: c = \text{sqr } (\text{tval } X) - \text{sNorm2 } (\text{sComponent } X)$
and $y5: y5 = (B \oplus ((-b/(2*a)) \otimes D))$
shows $(\text{qcase5 } a \ b \ c \longrightarrow S = \{y5\})$
 $\langle \text{proof} \rangle$

lemma *lemLineMeetsCone6*:
assumes $\neg (x \in l)$
and *isLine* l
and $S = l \cap \text{regularConeSet } x$
and $l: l = \text{line } B \ D$
and $X: X = (B \ominus x)$
and $a: a = \text{mNorm2 } D$
and $b: b = 2*(\text{tval } X)*(\text{tval } D) - 2*((\text{sComponent } D) \odot_s (\text{sComponent } X))$
and $c: c = \text{sqr } (\text{tval } X) - \text{sNorm2 } (\text{sComponent } X)$
and $ym: ym = (B \oplus (((-b - (\text{sqr } (\text{discriminant } a \ b \ c))) / (2*a)) \otimes D))$
and $yp: yp = (B \oplus (((-b + (\text{sqr } (\text{discriminant } a \ b \ c))) / (2*a)) \otimes D))$
shows $(\text{qcase6 } a \ b \ c \longrightarrow (ym \neq yp) \wedge S = \{ym, yp\})$
 $\langle \text{proof} \rangle$

lemma *lemConeLemma1*:
assumes $\neg (x \in l)$
and *isLine* l
and $S = l \cap \text{regularConeSet } x$
shows $\text{finite } S \wedge \text{card } S \leq 2$
 $\langle \text{proof} \rangle$

lemma *lemConeLemma2*:
assumes $\neg (\text{regularCone } x \ w)$
shows $\exists l. (\text{onLine } w \ l) \wedge (\neg (x \in l)) \wedge (\text{card } (l \cap (\text{regularConeSet } x)) = 2)$

$\langle proof \rangle$

lemma *lemLineInsideRegularConeHasFiniteSlope:*

assumes *insideRegularCone* $x\ p$

and $l = \text{lineJoining } x\ p$

shows *lineSlopeFinite* l

$\langle proof \rangle$

lemma *lemInvertibleOnMeet:*

assumes *invertible* f

and $S = A \cap B$

shows $\text{applyToSet } (\text{asFunc } f)\ S = (\text{applyToSet } (\text{asFunc } f)\ A) \cap$
 $(\text{applyToSet } (\text{asFunc } f)\ B)$

$\langle proof \rangle$

lemma *lemInsideCone:*

shows *insideRegularCone* $x\ p \longleftrightarrow$

$\neg(\text{vertex } x\ p \vee \text{outsideRegularCone } x\ p \vee \text{onRegularCone } x$
 $p)$

$\langle proof \rangle$

lemma *lemOnRegularConeIff:*

assumes $l = \text{lineJoining } x\ p$

shows *onRegularCone* $x\ p \longleftrightarrow (l \cap \text{regularConeSet } x = l)$

$\langle proof \rangle$

lemma *lemOutsideRegularConeImplies:*

shows *outsideRegularCone* $x\ p$

$\longrightarrow (\exists\ l\ p' . (p' \neq p) \wedge \text{onLine } p'\ l \wedge \text{onLine } p\ l$
 $\wedge (l \cap \text{regularConeSet } x = \{\}))$

$\langle proof \rangle$

lemma *lemTimelikeInsideCone:*

assumes *insideRegularCone* $x\ p$

shows *timelike* $(p \ominus x)$

$\langle proof \rangle$

end

end

31 ReverseCauchySchwarz

This theory defines and proves the "reverse" Cauchy-Schwarz inequality for timelike vectors in the Minkowski metric.

```
theory ReverseCauchySchwarz
  imports CauchySchwarz
begin
```

Rather than construct the proof, one could simply have asserted the claim as an axiom. We did this during development of the main proof, and then returned to this section later. In practice the axiom we chose to assert contained far more information than required, because we eventually found a proof that only required consideration of timelike vectors (our axiom considered lightlike vectors as well).

```
class ReverseCauchySchwarz = CauchySchwarz
```

```
begin
```

```
lemma lemTimelikeNotZeroTime:
  assumes timelike v
  shows tval v  $\neq$  0
   $\langle$ proof $\rangle$ 
```

```
lemma lemOrthogmToTimelike:
  assumes timelike u
and orthogm u v
and v  $\neq$  origin
shows spacelike v
   $\langle$ proof $\rangle$ 
```

```
lemma lemNormaliseTimelike:
  assumes timelike v
and s = sComponent ((1/tval v) $\otimes$ v)
shows (0  $\leq$  sNorm2 s < 1)  $\wedge$  (tval ((1/tval v) $\otimes$ v) = 1)
   $\langle$ proof $\rangle$ 
```

```
lemma lemReverseCauchySchwarz:
  assumes timelike X  $\wedge$  timelike D
shows sqr (X  $\odot$  m D)  $\geq$  (mNorm2 X)*(mNorm2 D)
   $\langle$ proof $\rangle$ 
```

```
end
```

end

32 KeyLemma

This theory establishes a "key lemma": if you draw a line through a point inside a cone, that line will intersect the cone in no fewer than 1 and no more than 2 points.

theory *KeyLemma*

imports *Classification ReverseCauchySchwarz*

begin

class *KeyLemma* = *Classification* + *ReverseCauchySchwarz*

begin

lemma *lemInsideRegularConeImplies:*

assumes *insideRegularCone x p*

and $D \neq \text{origin}$

and $l = \text{line } p \ D$

shows $0 < \text{card } (l \cap \text{regularConeSet } x) \leq 2$

<proof>

end

end

33 Cardinalities

For our purposes the only relevant cardinalities are 0, 1, 2 and more-than-2 (a proxy for "infinite"). We will use these cardinalities when looking at how lines intersect cones, using the size of the intersection set to characterise whether points are inside, on or outside of lightcones.

theory *Cardinalities*

imports *Functions*

begin

class *Cardinalities* = *Functions*

begin

lemma *lemInjectiveValueUnique:*

```

    assumes injective f
  and    isFunction f
  and    f x y
  shows  { q. f x q } = { y }
    <proof>

```

```

lemma lemBijectionOnTwo:
  assumes bijective f
  and    isFunction f
  and    s ⊆ domain f
  and    card s = 2
  shows  card (applyToSet f s) = 2
    <proof>

```

```

lemma lemElementsOfSet2:
  assumes card S = 2
  shows ∃ p q . (p ≠ q) ∧ p ∈ S ∧ q ∈ S
    <proof>

```

```

lemma lemThirdElementOfSet2:
  assumes (p ≠ q) ∧ p ∈ S ∧ q ∈ S ∧ (card S = 2)
  and    r ∈ S
  shows  p = r ∨ q = r
    <proof>

```

```

lemma lemSmallCardUnderInvertible:
  assumes invertible f
  and    0 < card S ≤ 2
  shows  card S = card (applyToSet (asFunc f) S)
    <proof>

```

```

lemma lemCardOfLineIsBig:
  assumes x ≠ p
  and    onLine x l ∧ onLine p l
  shows  ∃ p1 p2 p3 . (onLine p1 l ∧ onLine p2 l ∧ onLine p3 l)
    ∧ (p1 ≠ p2 ∧ p2 ≠ p3 ∧ p3 ≠ p1)
    <proof>

```

```

end
end

```

34 AffineConeLemma

This theory shows that affine approximations preserve "inside-ness" of points relative to cones.

```
theory AffineConeLemma
  imports KeyLemma TangentLineLemma Cardinalities
begin

class AffineConeLemma = KeyLemma + TangentLineLemma + Cardinalities
begin

lemma lemInverseOfAffInvertibleIsAffInvertible:
  assumes affInvertible A
and       $\forall x y . A\ x = y \longleftrightarrow A'\ y = x$ 
shows    affInvertible A'
   $\langle proof \rangle$ 

lemma lemInsideRegularConeUnderAffInvertible:
  assumes affInvertible A
and      insideRegularCone x p
and      regularConeSet (A x) = applyToSet (asFunc A) (regularConeSet x)
shows    insideRegularCone (A x) (A p)
   $\langle proof \rangle$ 

end
end
```

35 NoFTLGR

This theory completes the proof of NoFTLGR.

```
theory NoFTLGR
  imports ObserverConeLemma AffineConeLemma
begin

class NoFTLGR = ObserverConeLemma + AffineConeLemma
begin
```

The theorem says: if observer m encounters observer k (so that they are both present at the same spacetime point x), then k

is moving at sub-light speed relative to m . In other words, no observer ever encounters another observer who appears to be moving at or above lightspeed.

theorem *lemNoFTLGR*:

assumes $ass1: x \in wline\ m\ m \cap wline\ m\ k$
and $ass2: tl\ l\ m\ k\ x$
and $ass3: v \in lineVelocity\ l$
and $ass4: \exists\ p.\ (p \neq x) \wedge (p \in l)$
shows $(lineSlopeFinite\ l) \wedge (sNorm2\ v < 1)$
 $\langle proof \rangle$

end

end

References

- [1] M. Stannett and I. Németi. Using Isabelle/HOL to verify first-order relativity theory. *Journal of Automated Reasoning*, 52(4):361–378, 2014.
- [2] M. Stannett and I. Németi. No faster-than-light observers. *Archive of Formal Proofs*, April 2016. https://isa-afp.org/entries/No_FTL_observers.html, Formal proof development.