The Nash-Williams Theorem

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Abstract

In 1965, Nash-Williams [1] discovered a generalisation of the infinite form of Ramsey's theorem. Where the latter concerns infinite sets of n-element sets for some fixed n, the Nash-Williams theorem concerns infinite sets of finite sets (or lists) subject to a "no initial segment" condition. The present formalisation follows Todorčević [2].

Contents

1 The Pointwise Less-Than Relation Between Two Sets

theory *Nash-Extras*
 imports *HOL−Library.Ramsey HOL−Library.Countable-Set*

begin

definition *less-sets* :: $[{}'a::order\ set,\ {}'a::order\ set] \Rightarrow bool$ (**infixr** ‹≪› *50*)
 where $A \ll B \equiv \forall x{\in}A.\ \forall y{\in}B.\ x < y$

lemma *less-sets-empty*[*iff*]: $S \ll \{\}$ $\{\} \ll T$
 ⟨*proof*⟩

lemma *less-setsD*: $[\![A \ll B;\ a \in A;\ b \in B]\!] \Longrightarrow a < b$
 ⟨*proof*⟩

lemma *less-sets-irrefl* [*simp*]: $A \ll A \longleftrightarrow A = \{\}$
⟨*proof*⟩

lemma *less-sets-trans*: $[\![A \ll B; B \ll C; B \neq \{\}]\!] \Longrightarrow A \ll C$
⟨*proof*⟩

lemma *less-sets-weaken1*: $[\![A' \ll B; A \subseteq A']\!] \Longrightarrow A \ll B$
⟨*proof*⟩

lemma *less-sets-weaken2*: $[\![A \ll B'; B \subseteq B']\!] \Longrightarrow A \ll B$
⟨*proof*⟩

lemma *less-sets-imp-disjnt*: $A \ll B \Longrightarrow$ *disjnt* A B
⟨*proof*⟩

lemma *less-sets-UN1*: *less-sets* $(\bigcup \mathcal{A})$ $B \longleftrightarrow (\forall A \in \mathcal{A}.\ A \ll B)$
⟨*proof*⟩

lemma *less-sets-UN2*: *less-sets* A $(\bigcup \mathcal{B}) \longleftrightarrow (\forall B \in \mathcal{B}.\ A \ll B)$
⟨*proof*⟩

lemma *less-sets-Un1*: *less-sets* $(A \cup A')$ $B \longleftrightarrow A \ll B \wedge A' \ll B$
⟨*proof*⟩

lemma *less-sets-Un2*: *less-sets* A $(B \cup B') \longleftrightarrow A \ll B \wedge A \ll B'$
⟨*proof*⟩

lemma *strict-sorted-imp-less-sets*:
strict-sorted (*as* @ *bs*) $\Longrightarrow$ (*list.set as*) $\ll$ (*list.set bs*)
⟨*proof*⟩

lemma *Sup-nat-less-sets-singleton*:
fixes *n*::*nat*
assumes *Sup* $T < n$ *finite* T
shows *less-sets* T $\{n\}$
⟨*proof*⟩

end

2 The Nash-Williams Theorem

Following S. Todorčević, *Introduction to Ramsey Spaces*, Princeton University Press (2010), 11–12.

theory *Nash-Williams*
imports *Nash-Extras*
begin

lemma *finite-nat-Int-greaterThan-iff*:

fixes N :: *nat set*
shows *finite* $(N \cap \{n{<}..\}) \longleftrightarrow$ *finite* N
⟨*proof*⟩

2.1 Initial segments

definition *init-segment* :: *nat set* $\Rightarrow$ *nat set* $\Rightarrow$ *bool*
where *init-segment* $S\ T \equiv \exists\, S'.\ T = S \cup S' \wedge S \ll S'$

lemma *init-segment-subset*: *init-segment* $S\ T \Longrightarrow S \subseteq T$
⟨*proof*⟩

lemma *init-segment-refl*: *init-segment* $S\ S$
⟨*proof*⟩

lemma *init-segment-antisym*: $[\![$*init-segment* $S\ T$; *init-segment* $T\ S]\!] \Longrightarrow S{=}T$
⟨*proof*⟩

lemma *init-segment-trans*: $[\![$*init-segment* $S\ T$; *init-segment* $T\ U]\!] \Longrightarrow$ *init-segment* $S\ U$
⟨*proof*⟩

lemma *init-segment-empty2* [*iff*]: *init-segment* $S\ \{\} \longleftrightarrow S{=}\{\}$
⟨*proof*⟩

lemma *init-segment-Un*: $S \ll S' \Longrightarrow$ *init-segment* $S\ (S \cup S')$
⟨*proof*⟩

lemma *init-segment-iff0*:
shows *init-segment* $S\ T \longleftrightarrow S \subseteq T \wedge S \ll (T{-}S)$
⟨*proof*⟩

lemma *init-segment-iff*:
shows *init-segment* $S\ T \longleftrightarrow S{=}T \vee (\exists\, m \in T.\ S = \{n \in T.\ n < m\})$ (**is** *?lhs=?rhs*)
⟨*proof*⟩

lemma *init-segment-empty* [*iff*]: *init-segment* $\{\}\ S$
⟨*proof*⟩

lemma *init-segment-insert-iff*:
assumes *Sn*: $S \ll \{n\}$ **and** *TS*: $\bigwedge x.\ x \in T{-}S \Longrightarrow n{\le}x$
shows *init-segment* (*insert* $n\ S$) $T \longleftrightarrow$ *init-segment* $S\ T \wedge n \in T$
⟨*proof*⟩

lemma *init-segment-insert*:
assumes *init-segment* $S\ T$ **and** T: $T \ll \{n\}$
shows *init-segment* S (*insert* $n\ T$)
⟨*proof*⟩

2.2 Definitions and basic properties

definition *Ramsey* :: $[\mathit{nat\ set\ set}, \mathit{nat}] \Rightarrow \mathit{bool}$
where $\mathit{Ramsey}\ \mathcal{F}\ r \equiv \forall f \in \mathcal{F} \rightarrow \{..<r\}.$
$\forall M.\ \mathit{infinite}\ M \longrightarrow$
$(\exists N\ i.\ N \subseteq M \wedge \mathit{infinite}\ N \wedge i<r \wedge$
$(\forall j<r.\ j\neq i \longrightarrow f\ -`\ \{j\} \cap \mathcal{F} \cap \mathit{Pow}\ N = \{\}))$

Alternative, simpler definition suggested by a referee.

lemma *Ramsey-eq*:
$\mathit{Ramsey}\ \mathcal{F}\ r \longleftrightarrow (\forall f \in \mathcal{F} \rightarrow \{..<r\}.$
$\forall M.\ \mathit{infinite}\ M \longrightarrow$
$(\exists N\ i.\ N \subseteq M \wedge \mathit{infinite}\ N \wedge i<r \wedge \mathcal{F} \cap \mathit{Pow}\ N \subseteq f\ -`$
$\{i\}))$
$\langle \mathit{proof} \rangle$

definition *thin-set* :: $\mathit{nat\ set\ set} \Rightarrow \mathit{bool}$
where $\mathit{thin\text{-}set}\ \mathcal{F} \equiv \mathcal{F} \subseteq \mathit{Collect\ finite} \wedge (\forall S\in\mathcal{F}.\ \forall T\in\mathcal{F}.\ \mathit{init\text{-}segment}\ S\ T \longrightarrow$
$S=T)$

definition *comparables* :: $\mathit{nat\ set} \Rightarrow \mathit{nat\ set} \Rightarrow \mathit{nat\ set\ set}$
where $\mathit{comparables}\ S\ M \equiv \{T.\ \mathit{finite}\ T \wedge (\mathit{init\text{-}segment}\ T\ S \vee \mathit{init\text{-}segment}\ S\ T$
$\wedge\ T-S \subseteq M)\}$

lemma *comparables-iff*: $T \in \mathit{comparables}\ S\ M \longleftrightarrow \mathit{finite}\ T \wedge (\mathit{init\text{-}segment}\ T\ S$
$\vee\ \mathit{init\text{-}segment}\ S\ T \wedge T \subseteq S \cup M)$
$\langle \mathit{proof} \rangle$

lemma *comparables-subset*: $\bigcup(\mathit{comparables}\ S\ M) \subseteq S \cup M$
$\langle \mathit{proof} \rangle$

lemma *comparables-empty* [*simp*]: $\mathit{comparables}\ \{\}\ M = \mathit{Fpow}\ M$
$\langle \mathit{proof} \rangle$

lemma *comparables-mono*: $N \subseteq M \Longrightarrow \mathit{comparables}\ S\ N \subseteq \mathit{comparables}\ S\ M$
$\langle \mathit{proof} \rangle$

definition $\mathit{rejects}\ \mathcal{F}\ S\ M \equiv \mathit{comparables}\ S\ M \cap \mathcal{F} = \{\}$

abbreviation *accepts*
where $\mathit{accepts}\ \mathcal{F}\ S\ M \equiv \neg\ \mathit{rejects}\ \mathcal{F}\ S\ M$

definition *strongly-accepts*
where $\mathit{strongly\text{-}accepts}\ \mathcal{F}\ S\ M \equiv (\forall N\subseteq M.\ \mathit{rejects}\ \mathcal{F}\ S\ N \longrightarrow \mathit{finite}\ N)$

definition *decides*
where $\mathit{decides}\ \mathcal{F}\ S\ M \equiv \mathit{rejects}\ \mathcal{F}\ S\ M \vee \mathit{strongly\text{-}accepts}\ \mathcal{F}\ S\ M$

definition *decides-subsets*

where *decides-subsets* $\mathcal{F}$ $M \equiv \forall\, T.\ T \subseteq M \longrightarrow$ *finite* $T \longrightarrow$ *decides* $\mathcal{F}$ T M

lemma *strongly-accepts-imp-accepts*:
$[\![$*strongly-accepts* $\mathcal{F}$ S M; *infinite* $M]\!] \Longrightarrow$ *accepts* $\mathcal{F}$ S M
⟨*proof*⟩

lemma *rejects-trivial*: $[\![$*rejects* $\mathcal{F}$ S M; *thin-set* $\mathcal{F}$; *init-segment* F S; $F \in \mathcal{F}]\!] \Longrightarrow$ *False*
⟨*proof*⟩

lemma *rejects-subset*: $[\![$*rejects* $\mathcal{F}$ S M; $N \subseteq M]\!] \Longrightarrow$ *rejects* $\mathcal{F}$ S N
⟨*proof*⟩

lemma *strongly-accepts-subset*: $[\![$*strongly-accepts* $\mathcal{F}$ S M; $N \subseteq M]\!] \Longrightarrow$ *strongly-accepts* $\mathcal{F}$ S N
⟨*proof*⟩

lemma *decides-subset*: $[\![$*decides* $\mathcal{F}$ S M; $N \subseteq M]\!] \Longrightarrow$ *decides* $\mathcal{F}$ S N
⟨*proof*⟩

lemma *decides-subsets-subset*: $[\![$*decides-subsets* $\mathcal{F}$ M; $N \subseteq M]\!] \Longrightarrow$ *decides-subsets* $\mathcal{F}$ N
⟨*proof*⟩

lemma *rejects-empty* [*simp*]: *rejects* $\mathcal{F}$ $\{\}$ $M \longleftrightarrow$ *Fpow* $M \cap \mathcal{F} = \{\}$
⟨*proof*⟩

lemma *strongly-accepts-empty* [*simp*]: *strongly-accepts* $\mathcal{F}$ $\{\}$ $M \longleftrightarrow (\forall\, N{\subseteq}M.$ *Fpow* $N \cap \mathcal{F} = \{\} \longrightarrow$ *finite* $N)$
⟨*proof*⟩

lemma *ex-infinite-decides-1*:
assumes *infinite* M
obtains N **where** $N \subseteq M$ *infinite* N *decides* $\mathcal{F}$ S N
⟨*proof*⟩

proposition *ex-infinite-decides-finite*:
assumes *infinite* M *finite* S
obtains N **where** $N \subseteq M$ *infinite* N $\bigwedge T.\ T \subseteq S \Longrightarrow$ *decides* $\mathcal{F}$ T N
⟨*proof*⟩

lemma *sorted-wrt-subset*: $[\![X \in$ *list.set* l; *sorted-wrt* $(\leq)$ $l]\!] \Longrightarrow$ *hd* $l \subseteq X$
⟨*proof*⟩

Todorčević's Lemma 1.18

proposition *ex-infinite-decides-subsets*:
assumes *thin-set* $\mathcal{F}$ *infinite* M
obtains N **where** $N \subseteq M$ *infinite* N *decides-subsets* $\mathcal{F}$ N

⟨*proof*⟩

Todorčević's Lemma 1.19

proposition *strongly-accepts-1-19*:
assumes *acc*: *strongly-accepts* $\mathcal{F}$ *S M*
and *thin-set* $\mathcal{F}$ *infinite M* $S \subseteq M$ *finite S*
and *dsM*: *decides-subsets* $\mathcal{F}$ *M*
shows *finite* $\{n \in M.\ \neg$ *strongly-accepts* $\mathcal{F}$ (*insert n S*) $M\}$
⟨*proof*⟩

Much work is needed for this slight strengthening of the previous result!

proposition *strongly-accepts-1-19-plus*:
assumes *thin-set* $\mathcal{F}$ *infinite M*
and *dsM*: *decides-subsets* $\mathcal{F}$ *M*
obtains *N* **where** $N \subseteq M$ *infinite N*
$\bigwedge S\ n.\ [\![S \subseteq N;$ *finite S*; *strongly-accepts* $\mathcal{F}$ *S N*; $n \in N$; $S \ll \{n\}]\!]$
$\Longrightarrow$ *strongly-accepts* $\mathcal{F}$ (*insert n S*) *N*
⟨*proof*⟩

2.3 Main Theorem

lemma *Nash-Williams-1*: *Ramsey* $\mathcal{F}$ *1*
⟨*proof*⟩

theorem *Nash-Williams-2*:
assumes *thin-set* $\mathcal{F}$ **shows** *Ramsey* $\mathcal{F}$ *2*
⟨*proof*⟩

theorem *Nash-Williams*:
assumes $\mathcal{F}$: *thin-set* $\mathcal{F}$ $r > 0$ **shows** *Ramsey* $\mathcal{F}$ *r*
⟨*proof*⟩

end

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References

[1] C. S. J. A. Nash-Williams. On well-quasi-ordering transfinite sequences. *Mathematical Proceedings of the Cambridge Philosophical Society*, 61(1):33–39, 1965.

[2] S. Todorčević. *Introduction to Ramsey Spaces.* Princeton University Press, 2010.